Math 113 Homework 9

Not due (these are suggested problems only)

This assignment is a collection of practice problems for the most recent material, as well as some earlier problems in order to practice for the final. As usual, as you attempt to solve any of these problems, you should justify your solutions and/or answers with careful proofs.

Book problems: Axler Chapter 7 problems 16, 18, 19, 24, 26, 31, Chapter 10 problem 20.
Older problems for review: Chapter 6 problems 18, 21, 31, Chapter 8 problems 5, 8, 21, 22 (add in the phrase "... but whose characterstic polynomial equals $z^2(z-1)^2$" at the end), 30, Chapter 9 problems 12, 15.

1. *A formula for the determinant of* 3×3 *matrices.* Recall from class that the determinant $\det(T)$ of $T \in \mathcal{L}(V)$ is defined as the scalar in $\mathbb{F}$ such that

$$T(\mathbf{v}_1) \wedge \cdots \wedge T(\mathbf{v}_n) = \det(T) \cdot \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is any basis for V.

Suppose that $\dim V = 3$, and $\underline{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis for V. Let $T : V \to V$ be the linear operator defined by

$$\begin{aligned} T(\mathbf{v}_1) &= a\mathbf{v}_1 + d\mathbf{v}_2 + g\mathbf{v}_3 \\ T(\mathbf{v}_2) &= b\mathbf{v}_1 + e\mathbf{v}_2 + h\mathbf{v}_3 \\ T(\mathbf{v}_3) &= c\mathbf{v}_1 + f\mathbf{v}_2 + i\mathbf{v}_3. \end{aligned}$$

In other words, suppose the matrix of T with respect to $\underline{\mathbf{v}}$ is

$$\mathcal{M}(T, \underline{\mathbf{v}}) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Derive, using the definition we gave in class with exterior products, a formula for $\det(T)$ in terms of a, b, c, d, e, f, h, and i. (Hint: this formula should have six terms. It is easy to find the formula online, but you should be able to derive it in terms of our definition of determinants given in class).

2. *Properties of wedge products and determinants.* Let V be a vector space of dimension n.

(a) Prove that if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent in V, then $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k = 0$ in $\bigwedge^k V$.

(b) Let U be a subspace of dimension k, and assume that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is a basis of U. Prove that if $\mathbf{w}_1, \ldots, \mathbf{w}_k$ is another basis for U, then

$$\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k = a \cdot \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$$

for some non-zer scalar a.

(c) Now, suppose U, $(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ are as above, but that W is another k-dimensional subspace with basis $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$. Prove that if

$$\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k = a \cdot \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$$

for some nonzero scalar a, then $U = W$. (**Hint**: start with a basis of $U \cap W$, and extend it to a basis of $V \ldots$)

(d) Suppose $T : V \to V$ is a linear map, such that

$$V = U \oplus W$$

with U, W both T-invariant subspaces. Prove that

$$\det(T) = \det(T|_U) \cdot \det(T|_W).$$

(e) Suppose that U is a T-invariant subsace of a linear map $T : V \to V$. Let $\bar{T} : V/U \to V/U$ be the induced linear map on the quotient. Prove that

$$\det(T) = \det(T|_U) \cdot \det(\bar{T}).$$

(this generalizes the previous problem.)