Math 113 Final Exam

Thursday, June 11, 2013, 3.30 - 6.30pm.

Instructions. Answer the following problems carefully and completely. You must show all of your work, stating explicitly stating any result you are using, in order to receive full credit. You are welcome to use the results from the book, class or lecture notes, though no references, paper or digital, are permitted. Write your solutions in the provided blue books. Please return this examination, along with any scratch paper used, with your solution books.

If on a multi-part problem you cannot do a part (e.g., part (a)), you may still assume it is true for subsequent parts (e.g., part (b) or (c)) if it is helpful.

Name:

____

Stanford ID number:

____

Signature acknowledging the honor code:

____

1. (25 points total) Let $\mathcal{P}_2(\mathbb{R})$ denote the real vector space of polynomials of degree ≤ 2. Consider the following inner product on $\mathcal{P}_2(\mathbb{R})$:

$$\langle p, q \rangle := \frac{1}{\sqrt{2}} \int_{-1}^{1} p(x)q(x)dx$$

(a) (10 points) Use the Gram-Schmidt method to find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

(b) (5 points) Find an isomorphism $T : \mathcal{P}^2(\mathbb{R}) \xrightarrow{\sim} \mathbb{R}^3$ such that

$$\langle p, q \rangle = \langle Tp, Tq \rangle_{Eucl}, \tag{1}$$

where $\langle \cdot, \cdot \rangle_{Eucl}$ denotes the Euclidean dot product on $\mathbb{R}^3$ (you do not need to prove this is an isomorphism, but you will need to verify (1)).

(c) (10 points) Find the polynomial $p(x) \in \mathcal{P}_2(\mathbb{R})$ which best approximates the function $f(x) = x^3$ on $[-1, 1]$, in the sense that it minimizes $||p(x) - f(x)||$ (using the above inner product).

2. (35 points total; 7 points each) *Prove or disprove.* For each of the following statements, say whether the statement is True or False. Then, prove the statement if it is true, or disprove (find a counterexample with justification) if it is false. (Note: simply stating "True" or "False" will receive no credit).

(a) If V is an inner product space and $S : V \to V$ is an *isometry*, then $S^* = S$.

(b) If V is a finite-dimensional inner product space, and $T : V \to V$ is a map satisfying $T^* = p(T)$ for some polynomial $p(z)$, then $||T\mathbf{v}|| = ||T^*\mathbf{v}||$ for every $\mathbf{v} \in V$.

(c) If S and T are two nilpotent linear operators on a finite-dimensional vector space V, and $ST = TS$, then $S + T$ is nilpotent.

(d) If V is an inner product space and $T : V \to V$ is self-adjoint, then for any basis $\underline{\mathbf{v}} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of V the matrices of T and T^* with respect to $\underline{\mathbf{v}}$ are conjugate transposes, e.g.,

$$\mathcal{M}(T, \underline{\mathbf{v}}) = \overline{(\mathcal{M}(T^*, \underline{\mathbf{v}})^T}.$$

(e) If V is a finite-dimensional inner product space and $T : V \to W$ a linear map such that $T^* : W \to V$ is injective, then T is surjective.

3. (30 points total)

(a) (5 points) State the Jordan Normal Form theorem for linear maps $T : V \to V$, where V is a finite-dimensional complex vector space.

(b) (15 points) Suppose $T : \mathbb{C}^7 \to \mathbb{C}^7$ is a linear map with characteristic polynomial

$$q_T(z) = (z-2)^2(z-3)^2(z-4)^3.$$

Suppose also that $\dim\ker(T-2I) = \dim\ker(T-4I) = 1$ and $\dim\ker(T-3I) = 2$. Find a matrix which is a Jordan normal form for T. Be sure to justify your answer.

(c) (10 points) What is the minimal polynomial of T?

4. (20 points total; 10 points each) Suppose that $(V, \langle\cdot,\cdot\rangle)$ is a finite-dimensional inner product space, and $T : V \to V$ is a *positive linear operator* with all eigenvalues strictly greater than 0.

(a) (10 points) Prove that

$$\langle \mathbf{u}, \mathbf{v}\rangle' := \langle T\mathbf{u}, \mathbf{v}\rangle \tag{2}$$

defines a new inner product on V.

(b) (10 points) Suppose that T is as above, and $S : V \to V$ is a self-adjoint linear map (with respect to the original inner product $\langle\cdot,\cdot\rangle$. Prove that ST is diagonalizable (i.e. that it admits a basis of eigenvectors).

Hint/warning: ST is not self-adjoint with respect to the original inner product $\langle\cdot,\cdot\rangle$; in fact, if $*$ denotes taking adjoints with respect to the original inner product, note that $(ST)^* = T^*S^* = TS$, which is not in general equal to ST.

5. (15 points) *Determinants via wedge products.* Let V be a 4-dimensional real vector space. Suppose $T : V \to V$ is a linear map which, with respect to a basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ has the following form:

$$\mathbf{v}_1 \mapsto 3\mathbf{v}_2 - \mathbf{v}_3$$
$$\mathbf{v}_2 \mapsto 2\mathbf{v}_4$$
$$\mathbf{v}_3 \mapsto \mathbf{v}_1 + \mathbf{v}_4$$
$$\mathbf{v}_4 \mapsto \mathbf{v}_2 + \mathbf{v}_3$$

Using wedge products, calculate the determinant of T.

6. (15 points) *Norm bounds.* Let V be a finite-dimensional inner product space, and $T : V \to V$ an invertible linear map. Prove that there exists a positive real constant $c > 0$ such that

$$||T\mathbf{v}|| \geq c||\mathbf{v}||$$

for all vectors $\mathbf{v} \in V$.

Hint: Singular Value Decomposition or Polar Decomposition may be helpful.

7. (20 points total; 10 points each) *Composition as a map from the tensor product.* Let V be a finite-dimensional vector space of dimension n, and let $\mathcal{L}(V)$ denote the space of linear maps from V to V. Consider the map

$$comp : \mathcal{L}(V) \otimes \mathcal{L}(V) \longrightarrow \mathcal{L}(V)$$

defined on pure tensors by

$$S \otimes T \mapsto ST$$

and extended by linearity to a general element of the tensor product.

(a) Prove that *comp* is a linear map.

(b) Calculate $\dim \ker comp$.