Math 113 Final Exam: Solutions

Thursday, June 11, 2013, 3.30 - 6.30pm.

1. (25 points total) Let $\mathcal{P}_2(\mathbb{R})$ denote the real vector space of polynomials of degree ≤ 2. Consider the following inner product on $\mathcal{P}_2(\mathbb{R})$:

$$\langle p, q \rangle := \frac{1}{\sqrt{2}} \int_{-1}^{1} p(x)q(x)dx$$

(a) (10 points) Use the Gram-Schmidt method to find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

Solution: Denote $c = \frac{1}{\sqrt{2}}$. Let's begin with the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (1, x, x^2)$. We'll use the following facts: $\int_{-1}^{1} x^{2k+1} dx = 0$ for any odd number $2k+1$, as x^{2k+1} is an odd function, and $\int_{-1}^{1} x^{2k} dx = \frac{x^{2k+1}}{2k+1}|_{-1}^{1} = \frac{2}{2k+1}$.
In particular, this implies that $\langle x^2, x \rangle = \langle x, 1 \rangle = 0$.

Also, note that the norm $||1|| = \sqrt{\langle 1, 1 \rangle} = \sqrt{2/\sqrt{2}} = 2^{\frac{1}{4}}$.

- Step 1. Set $e_1 = 1/||1|| = k_1 \cdot 1$, where $k_1 = \frac{1}{2^{1/4}}$.
- Step 2. Set

$$\begin{aligned} f_2 &= \mathbf{v}_2 - \langle \mathbf{v}_2, e_1 \rangle \\ &= x - \langle x, k_1 \cdot 1 \rangle \cdot 1 \\ &= x \end{aligned}$$

because $\int_{-1}^{1} x dx = 0$. Now, normalize: first compute that $||f_2||^2 = \frac{1}{\sqrt{2}} \int_{-1}^{1} x^2 dx = \frac{\sqrt{2}}{3}$, so

$$e_2 = f_2/||f_2|| = k_2 x$$

where $k_2 = \frac{\sqrt{3}}{2^{1/4}}$. Finally,

- Step 3. Set

$$\begin{aligned}
f_3 &= \mathbf{v}_3 - \langle \mathbf{v}_3, e_2 \rangle \cdot e_2 - \langle \mathbf{v}_3, e_1 \rangle \cdot e_1 \\
&= x^2 - \langle x^2, k_2 x \rangle - \langle x^2, k_1 \cdot 1 \rangle \\
&= x^2 - k_1 \left(\int_{-1}^{1} x^2 dx \right) \cdot 1 \\
&= x^2 - 2k_1/3 \cdot 1
\end{aligned}$$

Then, normalize: first compute that

$$\begin{aligned}
||f_3||^2 &= c \int_{-1}^{1} x^4 dx - 4k_1/3c \int_{-1}^{1} x^2 dx + 4/9c \int_{-1}^{1} k_1^2 dx \\
&= 2c/5 - 8ck_1/9 + 4/9. \\
&= 2\sqrt{2}/5 - 8 \cdot 2^{1/4}/9 + 4/9.
\end{aligned}$$

Letting $1/k_3$ be the square root of this number, we set

$$e_3 = f_3/||f_3|| = k_3(x^2 - 2k_1/3 \cdot 1).$$

The result (e_1, e_2, e_3) is an orthonormal basis, by the Gram-Schmidt method.

Remarks: The Gram-Schmidt formula, properly applied (but not necessarily simplified, received full credit. A common mistake (only worth 1 or 2 points off) was to assume that the vector 1 had norm 1, which it does not.

(b) (5 points) Find an isomorphism $T : \mathcal{P}^2(\mathbb{R}) \xrightarrow{\sim} \mathbb{R}^3$ such that

$$\langle p, q \rangle = \langle Tp, Tq \rangle_{Eucl}, \tag{1}$$

where $\langle \cdot, \cdot \rangle_{Eucl}$ denotes the Euclidean dot product on $\mathbb{R}^3$ (you do not need to prove this is an isomorphism, but you will need to verify (1)).

Solution: Let (e_1, e_2, e_3) be as above. Then, define $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ by sending e_i to the ith standard basis vector e_i' (here we're using e_i' because the symbol e_i is already taken).

Note that T sends an orthonormal basis to an orthonormal basis, so by Axler Chapter 7, it's an isometry, which in particular implies that $\langle T\mathbf{u}, T\mathbf{v}\rangle_{Eucl} = \langle \mathbf{u}, \mathbf{v}\rangle$ for any pair of vectors $\mathbf{u}$, $\mathbf{v}$ (by the same lemma in Axler). Alternatively, one could directly check this.

(c) (10 points) Find the polynomial $p(x) \in \mathcal{P}_2(\mathbb{R})$ which best approximates the function $f(x) = x^3$ on $[-1, 1]$, in the sense that it minimizes $||p(x) - f(x)||$ (using the above inner product).

Solution: The answer is given by the *orthogonal projection* P_U of x^3 onto the subspace $U = \mathcal{P}_2(\mathbb{R})$ of $\mathcal{P}_3(\mathbb{R})$.

One way to compute this orthogonal projection is as

$$\langle x^3, e_1\rangle e_1 + \langle x^3, e_2\rangle e_2 + \langle x^3, e_3\rangle e_3.$$

$$\begin{aligned} &= 0 + (c\int_{-1}^{1} k_2 x^4 dx)(k_2 x) + 0 \\ &= (2ck_2^2/5)x \\ &= 2(\frac{1}{\sqrt{2}})(\frac{3}{\sqrt{2}})(\frac{1}{5})x \\ &= \frac{3}{5}x. \end{aligned}$$

where the first and last terms are zero because they are integrals of odd functions over the interval $[-1, 1]$.

Remarks: A common approach here was to simply write out the expression $||p(x) - x^3||^2$ for an arbitrary polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ and then to try to minimize this expression in a_1, a_2, and a_3. While such a method is certainly valid, it is much more computationally difficult (and as a result, very few such approaches succeeded). On the other hand, simply stating the Axler result that orthogonal projection provided the correct answer, along with a correct formula for orthogonal projection, was worth most of the points.

2. (35 points total; 7 points each) *Prove or disprove.* For each of the following statements, say whether the statement is True or False. Then, prove the statement if it is true, or disprove (find a counterexample with justification) if it is false. (Note: simply stating "True" or "False" will receive no credit).

(a) If V is an inner product space and $S : V \to V$ is an *isometry*, then $S^* = S$.

Solution: False. All that is guaranteed is that $S^* = S^{-1}$. For an example, consider the isometry $S : \mathbb{C} \to \mathbb{C}$ given by multiplication by i. Note that S^* is multiplication by $-i$, which is not equal to S.

(b) If V is a finite-dimensional inner product space, and $T : V \to V$ is a map satisfying $T^* = p(T)$ for some polynomial $p(z)$, then $||T\mathbf{v}|| = ||T^*\mathbf{v}||$ for every $\mathbf{v} \in V$.

Solution: True. We check first that T is *normal*: Note that $TT^* = Tp(T) = p(T)T = T^*T$ because any two polynomials in T commute. Then by a Lemma in Axler, normality of T is equivalent to $T\mathbf{v}$ and $T^*\mathbf{v}$ having the same norm for any $\mathbf{v}$.

(c) If S and T are two nilpotent linear operators on a finite-dimensional vector space V, and $ST = TS$, then $S + T$ is nilpotent.

Solution: True. Suppose that $S^k = 0$ and $T^l = 0$ for some k, l. Then, using binomial expansion,

$$(S+T)^{k+l} = \sum_{i=0}^{k+l} \binom{k+l}{i} S^i T^{k+l-i}$$

where we've critically used the fact that $ST = TS$ to equate expressions like $SSTS$ and $SSST$. But note that in each term of the form $S^i T^{k+l-i}$, if $i < k$, then $k + l - i > l$. Namely, either the exponent of S is greater than or equal to k or the exponent of T is greater than or equal to l. Thus, each term is 0.

(d) If V is an inner product space and $T : V \to V$ is self-adjoint, then for any basis $\underline{\mathbf{v}} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of V the matrices of T and T^* with respect to $\underline{\mathbf{v}}$ are conjugate transposes, e.g.,

$$\mathcal{M}(T, \underline{\mathbf{v}}) = \overline{\mathcal{M}(T^*, \underline{\mathbf{v}})^T}.$$

Solution: False, this is only necessarily true with respect to orthonormal bases. For a counterexample, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ send e_1 to e_2 and e_2 to e_1

(where e_i are the standard basis vectors). The matrix of T with respect to the orthonormal basis (e_1, e_2) is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is equal to its conjugate transpose, hence T is self-adjoint. However, with respect to the basis $\underline{\mathbf{v}} = (\mathbf{v}_1, \mathbf{v}_2) = (e_1 + e_2, e_2)$, T maps $\mathbf{v}_1$ to $\mathbf{v}_1$ and $\mathbf{v}_2$ to $\mathbf{v}_1 - \mathbf{v}_2$. Thus, the matrix of T with respect to this basis is

$$\mathcal{M}(T, \underline{\mathbf{v}}) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

which is not equal to its conjugate transpose.

(e) If V is a finite-dimensional inner product space and $T : V \to W$ a linear map such that $T^* : W \to V$ is injective, then T is surjective.

Solution: True. Note that by Axler, $\ker T^* = (\text{im } T)^\perp$. Now, if T^* is injective, then the former space is 0. This means $(\text{im } T)^\perp = 0$, which implies $\text{im } T = V$.

3. (30 points total)

(a) (5 points) State the Jordan Normal Form theorem for linear maps $T : V \to V$, where V is a finite-dimensional complex vector space.

Solution: If V is a finite-dimensional complex vector space and $T : V \to V$ is a linear operator then there exists a basis $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ of V, called a *Jordan basis*, such that the matrix of T with respect to $\underline{\mathbf{v}}$ is *block-diagonal*, i.e. of the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & A_k \end{pmatrix},$$

where each A_i is a square upper triangular matrix of the form

$$A_i = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & \vdots & \vdots & \vdots & 0 \\ 0 & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix},$$

where λ is some eigenvalue of T.

(b) (15 points) Suppose $T : \mathbb{C}^7 \to \mathbb{C}^7$ is a linear map with characteristic polynomial

$$q_T(z) = (z-2)^2(z-3)^2(z-4)^3.$$

Suppose also that $\dim\ker(T-2I) = \dim\ker(T-4I) = 1$ and $\dim\ker(T-3I) = 2$. Find a matrix which is a Jordan normal form for T. Be sure to justify your answer.

Solution: By Axler, this form for the characteristic polynomial implies that $\mathbb{C}^7$ decomposes as $U_2 \oplus U_3 \oplus U_4$ where U_λ denotes the generalized λ-eigenspace of T. Moreover, by Axler, the exponent of $(z-\lambda)$ in the characteristic polynomial is the dimension of U_λ, so $\dim U_2 = \dim U_3 = 2$, and $\dim U_4 = 3$.

Let W_λ denote the λ-eigenspace of T, so $W_\lambda \subset U_\lambda$. By JNF, there exists a Jordan basis for T, with associated matrix a collection of Jordan blocks. We claim that *the number of Jordan blocks with a given λ on the diagonal is in fact the dimension of the eigenspace W_λ*. Namely, for each Jordan block, the basis vector corresponding to the top left corner of that Jordan block is a genuine eigenvalue. Moreover, by explicit computation, it is the only basis vector within that given Jordan block in the kernel of $T-\lambda I$. More explicitly, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a substring of a basis of V corresponding to a single $k \times k$ Jordan block with λ on the diagonal, then on this collection of vectors, $T-\lambda I$, whose corresponding matrix is the same sized Jordan block with 0's on the diagonal, acts as

$$\begin{aligned}
\mathbf{v}_1 &\mapsto 0 \\
\mathbf{v}_2 &\mapsto 1 \\
&\vdots \\
\mathbf{v}_k &\mapsto \mathbf{v}_{k-1}
\end{aligned}$$

and thus $T - \lambda I$ has one-dimensional kernel on $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. On all of V, the kernel of $T - \lambda I$ is the direct sum of the kernels of $T - \lambda I$ on each subspace corresponding to a λ-Jordan block, each of which is one dimensional. Thus, the total dimension of $\ker(T - \lambda I)$ is the number of Jordan blocks.

Let us now examine the which Jordan blocks must appear in a Jordan matrix for T for each eigenvalue λ. First consider the case $\lambda = 2$ or 4, in which case $\dim W_\lambda = 1$. Using the above claim, we conclude there can be only one Jordan block for λ For $\lambda = 2$, since $\dim U_2 = 2$, we conclude there is one 2 by 2 Jordan block with 2 the diagonal. Similarly, since $\dim U_4 = 3$, we conclude that there is one 3 by 3 Jordan block with 4 on the diagonal.

Now, for $\lambda = 3$, since $\dim W_3 = 2$, there must be 2 Jordan blocks with 3's on the diagonal. Since the total dimension of U_3, the sum of sizes of all 3-Jordan blocks, is 3, both eigenvalue 3 blocks must be 1 by 1.

Thus, up to reordering the Jordan blocks, the Jordan matrix for T looks like

$$\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}.$$

Remarks: A number of students thought $\dim \ker(T - \lambda I)$ was the maximal size of a Jordan block, which led to an incorrect answer on this section. As a simple example to see why this is not the case, consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix with respect to some basis is

$$3I = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Note that this matrix is a collection of 3 1 by 1 Jordan blocks, so the maximal size of a Jordan block is 1. However, the dimension of the 3 eigenspace $\ker(T - 3I)$ is 3.

Also, it was possible to prove this problem without verifying the entire claim, just by observing that if there are e.g., k λ-Jordan blocks, then there are at least k linearly independent λ-eigenvectors (then use process of elimination).

(c) (10 points) What is the minimal polynomial of T?

Solution. The minimal polynomial is

$$\prod_{\lambda \text{ eigenvalue of } T} (z - \lambda)^{k_\lambda}$$

where k_λ is the size of the largest Jordan block with λ on the diagonal for any Jordan matrix for T. Using the Jordan form determined in the previous section, we can read off that the minimal polynomial is

$$(z - 2)^2(z - 3)(z - 4)^3.$$

4. (20 points total; 10 points each) Suppose that $(V, \langle \cdot, \cdot \rangle)$ is a finite-dimensional inner product space, and $T : V \to V$ is a *positive linear operator* with all eigenvalues strictly greater than 0.

(a) (10 points) Prove that

$$\langle \mathbf{u}, \mathbf{v} \rangle' := \langle T\mathbf{u}, \mathbf{v} \rangle \tag{2}$$

defines a new inner product on V.

Solution: Note first that since T is positive with all eigenvalues strictly greater than zero, there exists a unique square root P which is positive with all eigenvalues strictly greater than zero. Then

$$\langle T\mathbf{u}, \mathbf{v}\rangle = \langle PP\mathbf{u}, \mathbf{v}\rangle = \langle P\mathbf{u}, P\mathbf{v}\rangle.$$

with P positive and all eigenvalues > 0.

Now, we can quickly verify the conditions of being an inner product (it's not strictly necessary to do so, but our solution will use P):

- positivity: Note that $\langle \mathbf{v}, \mathbf{v}\rangle' = \langle P\mathbf{v}, P\mathbf{v}\rangle = ||P\mathbf{v}||^2 \geq 0$. (alternatively, this follows by definition of T being positive).
- definiteness: Since P is positive, it is self-adjoint so admits an orthonormal eigenbasis $e_1, \ldots, e_n$. Let $(\lambda_1, \ldots, \lambda_n)$ be the eigenvalues; these are strictly greater than zero by Axler. Then note that any non-zero $\mathbf{v} = \sum a_i e_i$,

  $$||P\mathbf{v}||^2 = \sum a_i \lambda_i^2$$

  which is strictly positive as each λ_i is non-zero and positive, and at least one a_i is not equal to zero.
- linear in first slot: This follows from linearity of P, and linearity in the first slot of the original inner product.
- Conjugate symmetry: this follows from conjugate symmetry of the original inner product, and either self-adjointness of T or the symmetry of the expression $\langle P\mathbf{u}, P\mathbf{v}\rangle$.

Remarks: It was not necessary to take a square root to prove that $\langle \cdot, \cdot\rangle'$ is an inner product; one can simply use self-adjointness of T, the fact that T is linear, the positivity condition, the fact that all eigenvalues were strictly positive, and the fact that $\langle \cdot, \cdot\rangle$ is an inner product to obtain the above properties for $\langle \cdot, \cdot\rangle'$. This exercise did not require using any bases, although some students successfully took this approach.

(b) (10 points) Suppose that T is as above, and $S : V \to V$ is a self-adjoint linear map (with respect to the original inner product $\langle \cdot, \cdot\rangle$. Prove that ST is diagonalizable (i.e. that it admits a basis of eigenvectors).

Hint/warning: ST is not self-adjoint with respect to the original inner product $\langle\cdot,\cdot\rangle$; in fact, if $*$ denotes taking adjoints with respect to the original inner product, note that $(ST)^* = T^*S^* = TS$, which is not in general equal to ST.

Solution: If ST is self-adjoint with respect to the modified inner product $\langle\cdot,\cdot\rangle'$, then by the spectral theorem, ST will have an orthonormal eigenbasis (with respect to the modified inner product). This in particular would imply that ST has an eigenbasis, as desired. So it suffices to verify that ST is self-adjoint with respect to $\langle\cdot,\cdot\rangle'$, which follows from a computation:

$$\begin{aligned}\langle ST\mathbf{u},\mathbf{v}\rangle' &= \langle T(ST\mathbf{u}),\mathbf{v}\rangle\\ &= \langle TST\mathbf{u},\mathbf{v}\rangle\\ &= \langle T\mathbf{u},S^*T^*\mathbf{v}\rangle\\ &= \langle T\mathbf{u},ST\mathbf{v}\rangle\\ &= \langle\mathbf{u},\mathbf{v}\rangle'\end{aligned}$$

where the second last equality followed from S and T individually being self-adjoint.

5. (15 points) *Determinants via wedge products.* Let V be a 4-dimensional real vector space. Suppose $T : V \to V$ is a linear map which, with respect to a basis $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)$ has the following form:

$$\begin{aligned}\mathbf{v}_1 &\mapsto 3\mathbf{v}_2 - \mathbf{v}_3\\ \mathbf{v}_2 &\mapsto 2\mathbf{v}_4\\ \mathbf{v}_3 &\mapsto \mathbf{v}_1 + \mathbf{v}_4\\ \mathbf{v}_4 &\mapsto \mathbf{v}_2 + \mathbf{v}_3\end{aligned}$$

Using wedge products, calculate the determinant of T.

Solution: We compute:

$$\begin{aligned}
T_*(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_4) &= T\mathbf{v}_1 \wedge T\mathbf{v}_2 \wedge T\mathbf{v}_3 \wedge T\mathbf{v}_4 \\
&= (3\mathbf{v}_2 - \mathbf{v}_3) \wedge (2\mathbf{v}_4) \wedge (\mathbf{v}_1 + \mathbf{v}_4) \wedge (\mathbf{v}_2 + \mathbf{v}_3) \\
&= (3\mathbf{v}_2 - \mathbf{v}_3) \wedge (2\mathbf{v}_4) \wedge \mathbf{v}_1 \wedge (\mathbf{v}_2 + \mathbf{v}_3) \\
&= 3\mathbf{v}_2 \wedge (2\mathbf{v}_4) \wedge \mathbf{v}_1 \wedge \mathbf{v}_3 + (-\mathbf{v}_3) \wedge (2\mathbf{v}_4) \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \\
&= 6\mathbf{v}_2 \wedge \mathbf{v}_4 \wedge \mathbf{v}_1 \wedge \mathbf{v}_3 - 2\mathbf{v}_3 \wedge \mathbf{v}_4 \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \qquad (3)
\end{aligned}$$

where we have expanded using multilinearity and canceled using the alternating condition. Now, note that

$$\begin{aligned}
\mathbf{v}_2 \wedge \mathbf{v}_4 \wedge \mathbf{v}_1 \wedge \mathbf{v}_3 &= -\mathbf{v}_4 \wedge \mathbf{v}_2 \wedge \mathbf{v}_1 \wedge \mathbf{v}_3 \\
&= \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_4 \wedge \mathbf{v}_3 \\
&= -\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \wedge \mathbf{v}_4
\end{aligned}$$

where we've swapped vectors one at a time with sign flips. Similarly,

$$\begin{aligned}
-\mathbf{v}_3 \wedge \mathbf{v}_4 \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 &= \mathbf{v}_2 \wedge \mathbf{v}_4 \wedge \mathbf{v}_1 \wedge \mathbf{v}_3 \\
&= -\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \wedge \mathbf{v}_4.
\end{aligned}$$

So the above expression becomes

$$(-6-2)\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \wedge \mathbf{v}_4 = (-8)\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \wedge \mathbf{v}_4,$$

hence $\det(T) = -8$.

6. (15 points) *Norm bounds*. Let V be a finite-dimensional inner product space, and $T : V \to V$ an invertible linear map. Prove that there exists a positive real constant $c > 0$ such that

$$||T\mathbf{v}|| \geq c||\mathbf{v}||$$

for all vectors $\mathbf{v} \in V$.

Hint: Singular Value Decomposition or Polar Decomposition may be helpful.

Solution: As indicated during the exam, SVD is probably the simpler way to proceed. By SVD, for any T, there exists a pair of orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V, and non-negative real numbers $s_1, \ldots, s_n$, called the *singular values* of T, such that

$$Te_i = s_i f_i.$$

Note first that if T is invertible, then T is injective, so each s_i is strictly positive. (if some $s_i = 0$, then $Te_i = 0f_i = 0$, a contradiction to injectivity).

Set $c = \min(s_1, \ldots, s_n) > 0$. Then, note that for any $\mathbf{v} \in V$, $\mathbf{v} = a_1e_1 + \cdots + a_ne_n$,

$$\begin{aligned} ||T\mathbf{v}||^2 &= ||a_1s_1f_1 + \cdots + a_ns_nf_n||^2 \\ &= \sum_{i=1}^{n} a_i^2 s_i^2 \\ &\geq c^2 \sum_{i=1}^{n} a_i^2 \\ &= c^2||\mathbf{v}||^2, \end{aligned}$$

where second and fourth lines used the pythagorean theorem, and the third line used the facts that every term in the sum was positive and each $s_i \geq c$ by definition. Taking square roots implies the desired result.

7. (20 points total; 10 points each) *Composition as a map from the tensor product.* Let V be a finite-dimensional vector space of dimension n, and let $\mathcal{L}(V)$ denote the space of linear maps from V to V. Consider the map

$$comp : \mathcal{L}(V) \otimes \mathcal{L}(V) \longrightarrow \mathcal{L}(V)$$

defined on pure tensors by

$$S \otimes T \mapsto ST$$

and extended by linearity to a general element of the tensor product.

(a) Prove that *comp* is a linear map.

Solution: By our class notes on the tensor product, it suffices to show that the induced map on pure tensors is *bilinear*; that is the map

$$\hat{comp} : \mathcal{L}(V) \times \mathcal{L}(V) \to \mathcal{L}(V)$$
$$(S, T) \mapsto comp(S \otimes T) = ST$$

is bilinear. We check that

$$\begin{aligned} \hat{comp}(aS + bS', T) &= (aS + bS')T \\ &= aST + bS'T \\ &= a \cdot \hat{comp}(S, T) + b \cdot \hat{comp}(S', T), \end{aligned}$$

verifying linearity in the first slot. Similarity,

$$\begin{aligned} \hat{comp}(S, aT + bT') &= S(aT + bT') \\ &= aST + bST' \\ &= a \cdot \hat{comp}(S, T) + b \cdot \hat{comp}(S, T'), \end{aligned}$$

verifying linearity in the second slot. (Both of these verifications use properties of composition which were discussed in class and Axler).

(b) Calculate $\dim \ker comp$.

Solution: First, by Axler we know that $\dim \mathcal{L}(V) = n^2$, and by class notes we know that $\dim \mathcal{L}(V) \otimes \mathcal{L}(V) = n^2 \cdot n^2 = n^4$.

Now, for any $S \in \mathcal{L}(V)$, the pure tensor $S \otimes I$ maps to S via *comp*; hence *comp* is surjective; i.e.

$$\dim \operatorname{im}\, comp = n^2.$$

Now, we can apply Rank-Nullity:

$$\begin{aligned} \dim \ker comp &= \dim(\mathcal{L}(V) \otimes \mathcal{L}(V)) - \dim \operatorname{im}\, comp \\ &= n^4 - n^2. \end{aligned}$$