Math 113 Midterm Exam

Thursday, May 7, 2013, 7 - 9 pm.

Instructions. Answer the following problems carefully and completely. You must show all of your work, stating explicitly stating any result you are using, in order to receive full credit. You are welcome to use the results from the book or class, though no references, paper or digital, are permitted. Write your solutions in the provided blue books. Please return this examination, along with any scratch paper used, with your solution books.

If on a multi-part problem you cannot do a part (e.g., part (a)), you may still assume it is true for subsequent parts (e.g., part (b) or (c)) if it is helpful.

Name:

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Stanford ID number:

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Signature acknowledging the honor code:

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1. (10 points) Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ be a linear operator. Suppose that there is a non-zero vector $\mathbf{v} \in V$ such that $T^3\mathbf{v} = T\mathbf{v}$. Show that at least one of the numbers 0, 1, -1 is an eigenvalue of T.

2. (35 points total, 7 points each) *Prove or disprove.* For each of the following statements, say whether the statement is True or False. Then, prove the statement if it is true, or disprove (find a counterexample with justification) if it is false. (Note: simply stating "True" or "False" will receive no credit).

(a) If the only eigenvalue of a linear map $T : V \to V$ is 0, then T is the zero map.

(b) Let V be a vector space of dimension n, and $\mathcal{P}_{n-1}(\mathbb{F})$ denote polynomials with degree less than or equal to $n-1$. Then, $\mathcal{L}(V, \mathbb{F})$ is isomorphic to $\mathcal{P}_{n-1}(\mathbb{F})$.

(c) Let U and V be 4-dimensional subspaces of $\mathbb{R}^7$. Then U and V contain a common non-zero vector.

(d) Let $T : V \to V$ be a linear transformation. Then, V is the direct sum of its subspaces $\ker T$ and $\mathrm{im} T$. That is, $V = \ker T \oplus \mathrm{im} T$.

(e) Let V be finite-dimensional and let $S, T : V \to V$ satisfy $ST = \mathbf{0}_V$. Then $\dim \mathrm{im}(S) + \dim \mathrm{im}(T) \leq \dim(V)$.

3. (25 points total) *Annihilators and quotients.* Let V be a vector space over $\mathbb{F}$. Recall that we defined the *dual space* V^* to be the vector space of linear maps from V to $\mathbb{F}$,

$$V^* = \mathcal{L}(V, \mathbb{F}).$$

Now, let $S \subset V$ be a subset. Define the *annihilator of* S to be the following subspace of V^*:

$$\mathrm{Ann}(S) := \{T : V \to \mathbb{F} \mid T(s) = \mathbf{0} \text{ for all } s \in S\}.$$

In words, $\mathrm{Ann}(S)$ is the subset of linear functionals that *annihilate all of* S, i.e. those functionals that send every element of S to $\mathbf{0}$.

(a) (10 points) Prove that $\operatorname{Ann}(S)$ is a subspace of V^*, and that

$$\operatorname{Ann}(S) = \operatorname{Ann}(\operatorname{span}(S)).$$

(b) (10 points) Now, let U and W be two subspaces of a V. Prove that $\operatorname{Ann}(U + W) = \operatorname{Ann}(U) \cap \operatorname{Ann}(W)$.

(c) (5 points) If W is a subspace of V, let V/W denote the quotient of V by W and let $(V/W)^*$ denote its linear dual. Construct a *canonical* (not using a basis) linear map

$$f_W : \operatorname{Ann}(W) \longrightarrow (\mathrm{V/W})^*.$$

other than the zero map (in fact, this map should be invertible). Also, define the inverse f_W^{-1} of this map (though you do not need to carefully check that $f_W \circ f_W^{-1} = I$ and $f_W^{-1} \circ f_W = I$.)

4. (20 points total, 10 points each) *Linear maps, kernels, and images.*

(a) Let $T : V \to W$ and $S : W \to U$ be linear maps between finite-dimensional vector spaces. Prove that if T is surjective, then

$$\dim(\ker ST) = \dim(\ker S) + \dim(\ker T).$$

(b) Now, suppose suppose $S, T \in \mathcal{L}(V)$, and V is finite-dimensional. Prove that $\ker S \subset \ker T$ if and only if there exists a linear map $R \in \mathcal{L}(V)$ with $T = RS$.

5. (15 points) Let V denote the subspace of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ spanned by the functions $\cos x$, $\sin x$, $x \cos x$, and $x \sin x$. Let $T : V \to V$ be the linear map defined by

$$(Tf)(x) := f(x + \pi) + f(-x).$$

Find an eigenbasis of V with respect to the linear map T. What is the matrix of T with respect to this basis? (Don't forget: $\sin(x+\pi) = -\sin x$, and $\cos(x+\pi) = -\cos x$).

6. (15 points) *An invariant subspace.* Let V be a vector space with dimension n. Suppose $T : V \to V$ is a linear transformation with n *distinct* eigenvalues $\lambda_1, \ldots, \lambda_n$

and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Set

$$\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_n$$

and consider the subspace W spanned by the vectors $\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, T^3\mathbf{v}, \ldots\}$ (W is the smallest T-invariant subspace containing $\mathbf{v}$). Prove that $W = V$.

7. (20 points total) *Matrices and polynomials.*

(a) (5 points) Let V and W be finite-dimensional vector spaces over $\mathbb{F}$, and $T : V \to W$ be a linear map. Give a definition of the matrix of T with respect to a pair of bases (one for V and one for W).

(b) (15 points) Let $V = \mathcal{P}_2(\mathbb{R})$ denote the vector space of polynomials of degree ≤ 2 with real coefficients. Consider the standard basis

$$\underline{\mathbf{v}} = (\mathbf{v}_1 = 1, \mathbf{v}_2 = x, \mathbf{v}_3 = x^2)$$

of $\mathcal{P}_2(\mathbb{R})$. On homework, you showed that the dual vector space $V^* = \mathcal{L}(V, \mathbb{R})$ has an induced *dual basis*

$$\underline{\mathbf{v}}^* := (\mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*),$$

where $\mathbf{v}_i^*$ is the linear map from V to $\mathbb{R}$ defined by

$$\mathbf{v}_i^*(\mathbf{v}_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i. \end{cases}$$

Now, let $T : V \to V^*$ be the linear map defined by

$$p \mapsto f_p$$

where $f_p \in V^* = (\mathcal{P}_2(\mathbb{R}))^*$ is the functional defined by

$$f_p(q) := \int_{-1}^{1} p(x)q(x)dx$$

(You do not need to prove this is a linear map). Determine the matrix of T with respect to the bases $\underline{\mathbf{v}}$, $\underline{\mathbf{v}}^*$.