Math 113 Midterm Exam—Solutions

Held Thursday, May 7, 2013, 7 - 9 pm.

1. (10 points) Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ be a linear operator. Suppose that there is a non-zero vector $\mathbf{v} \in V$ such that $T^3\mathbf{v} = T\mathbf{v}$. Show that at least one of the numbers 0, 1, -1 is an eigenvalue of T.

Solution: We have that $T^3\mathbf{v} = T\mathbf{v}$ for some nonzero vector $\mathbf{v}$. Therefore $(T^3 - T)\mathbf{v} = 0$. Powers of the linear map T commute with one another, so we may rewrite this as $T(T-I)(T+I)\mathbf{v} = 0$. Now, there are several cases:

Case 1: $\mathbf{v} \in \ker(T+I)$:

Then $\mathbf{v}$ is a nonzero -1–eigenvector of T.

Case 2: $\mathbf{v} \notin \ker(T+I)$, so $(T+I)\mathbf{v} \neq 0$, but $(T+I)\mathbf{v} \in \ker(T-I)$:

Then $(T+I)\mathbf{v}$ is a nonzero 1–eigenvector of T.

Case 3: $(T+I)\mathbf{v} \notin \ker(T-I)$:

Then $(T-I)(T+I)\mathbf{v}$ is not the zero vector. But $T(T-I)(T+I)\mathbf{v} = 0$, so $(T-I)(T+I)\mathbf{v}$ is a nonzero 0–eigenvector of T.

Therefore we have a nonzero eigenvector of T with eigenvector -1, 1 or 0, so at least one of these must be an eigenvalue of T.

Alternatively, it was acceptable to directly note that because $T(T-I)(T+I)\mathbf{v} = 0$, one of T, $T-I$ or $T+I$ must not be injective (because in order to have this equality with $\mathbf{v}$ non-zero, at some point, one of the operators T, $T-I$, or $T+I$ must send a non-zero vector to zero).

Common mistakes:

- Somehow using $T^3\mathbf{v} = T\mathbf{v}$ for all $\mathbf{v}$, rather than just for one fixed $\mathbf{v}$.
- Thinking that $T\mathbf{v} = 0$ for some $\mathbf{v}$ meant that T was the zero operator.
- Similarly, thinking that $T\mathbf{v} = \mathbf{v}$ for some $\mathbf{v}$ meant that T was the identity operator.

- Proving the converse – that if T has an eigenvalue of 1, -1 or 0, then there is a nonzero vector $\mathbf{v}$ with $T^3\mathbf{v} = T\mathbf{v}$.
- Getting $T(T - I)(T + I)\mathbf{v} = 0$, and then claiming that $\mathbf{v}$ was in either $\ker(T), \ker(T + I)$ or $\ker(T - I)$.

2. (35 points total, 7 points each) *Prove or disprove.* For each of the following statements, say whether the statement is True or False. Then, prove the statement if it is true, or disprove (find a counterexample with justification) if it is false. (Note: simply stating "True" or "False" will receive no credit).

(a) If the only eigenvalue of a linear map $T : V \to V$ is 0, then T is the zero map.

Solution: *False.* Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ sending $(x, y) \mapsto (y, 0)$. As we discussed in class, there is only one eigenvalue, 0, but clearly T is not the zero map.

More generally, if T is any linear map whose matrix for some basis is upper triangular, with all 0's on the diagonal, then in class (and in Axler) we showed that the only eigenvalue of T is 0. However, this matrix corresponding to T could have arbitrary entries above the diagonal, and thus need not be the zero matrix.

(b) Let V be a vector space of dimension n, and $\mathcal{P}_{n-1}(\mathbb{F})$ denote polynomials with degree less than or equal to $n - 1$. Then, $\mathcal{L}(V, \mathbb{F})$ is isomorphic to $\mathcal{P}_{n-1}(\mathbb{F})$.

Solution: *True.* Note that $V^* = \mathcal{L}(V, \mathbb{F})$ has the same dimension of V, which is n (either by a homework problem or the textbook), and $\mathcal{P}_{n-1}(\mathbb{F})$, which has standard basis $(1, x, \ldots, x^{n-1})$ also has dimension n. And any two finite-dimensional vector spaces with the same dimension are isomorphic, by results from class.

(c) Let U and V be 4-dimensional subspaces of $\mathbb{R}^7$. Then U and V contain a common non-zero vector.

Solution: *True*. By the dimension formula from class/Axler,

$$\dim(U+V) = \dim U + \dim V - \dim(U \cap V) = 4 + 4 - \dim(U \cap V)$$

but $U + V$ is a subspace of $\mathbb{R}^7$, so $\dim(U+V) \leq 7$. We conclude that $\dim(U \cap V) \geq 1$, so in particular $U \cap V$ contains a non-zero vector.

(d) Let $T : V \to V$ be a linear transformation. Then, V is the direct sum of its subspaces $\ker T$ and $\mathrm{im} T$. That is, $V = \ker T \oplus \mathrm{im} T$.

Solution: *False*. Consider the same map as part (a), $T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (y, 0)$. Note that $\ker T = \{(x, 0)\}$ and $\mathrm{im}(T) = \{(x, 0)\}$. In particular, $\ker T + \mathrm{im}(T) = \{(x, 0)\}$, which is not equal to all of $\mathbb{R}^2$.

(e) Let V be finite-dimensional and let $S, T : V \to V$ satisfy $ST = \mathbf{0}_V$. Then $\dim \mathrm{im}(S) + \dim \mathrm{im}(T) \leq \dim(V)$.

Solution: *True*. First note that $ST = \mathbf{0}_V$ implies that $\mathrm{im}(T) \subset \ker S$, so

$$\dim \mathrm{im}(T) \leq \dim \ker(S). \tag{1}$$

Now, by the rank-nullity theorem

$$\dim \ker(S) + \dim \mathrm{im}(S) = \dim V,$$

which together with (1) implies the desired statement.

3. (25 points total) *Annihilators and quotients*. Let V be a vector space over $\mathbb{F}$. Recall that we defined the *dual space* V^* to be the vector space of linear maps from V to $\mathbb{F}$,

$$V^* = \mathcal{L}(V, \mathbb{F}).$$

Now, let $S \subset V$ be a subset. Define the *annihilator of* S to be the following subspace of V^*:

$$\mathrm{Ann}(S) := \{T : V \to \mathbb{F} \mid T(s) = \mathbf{0} \text{ for all } s \in S\}.$$

In words, $\mathrm{Ann}(S)$ is the subset of linear functionals that *annihilate all of* S, i.e. those functionals that send every element of S to $\mathbf{0}$.

(a) (10 points) Prove that $\text{Ann}(S)$ is a subspace of V^*, and that

$$\text{Ann}(S) = \text{Ann}(\text{span}(S)).$$

Solution: By definition, $\text{Ann}(S)$ is a subset of V^*. We simply need to prove it's a subspace by verifying the following assertions:

- $\mathbf{0} \in \text{Ann}(S)$. This is because $\mathbf{0}(s) = 0$ for every $s \in S$ (indeed $\mathbf{0}$ applied to any vector is 0).
- Additive closure: Say $f, g \in \text{Ann}(S)$. Then, for any $s \in S$, note that $(f + g)(s) = f(s) + g(s) = 0 + 0 = 0$, so $f + g \in \text{Ann}(S)$.
- Closure under scalar multiplication: Say $f \in \text{Ann}(S)$. Then, for any scalar c, note that $(cf)(s) = c \cdot f(s) = c \cdot 0 = 0$, so $cf \in \text{Ann}(S)$.

Now, let's show $\text{Ann}(S) = \text{Ann}(\text{span})(S)$. Note clearly that $\text{Ann}(\text{span}(S)) \subset \text{Ann}(S)$, because any functional that annihilates all of $\text{span}(S)$ certainly annihilates S, which is a subset of $\text{span}(S)$. So we need to show the other inclusion: $\text{Ann}(S) \subset \text{Ann}(\text{span}(S))$. Let $f \in \text{Ann}(S)$. Then, consider any element $\mathbf{v}$ of $\text{span}(S)$, which can be written as a linear combination

$$a_0 s_0 + \cdots a_k s_k$$

of elements of S. We check that, since f is linear,

$$f(a_0 s_0 + \cdots + a_k s_k) = a_0 f(s_0) + \cdots + a_k f(s_k) = 0,$$

as $f(s_i) = 0$ for all $s_i \in S$. Thus, $f(\mathbf{v}) = 0$ for any $\mathbf{v} \in \text{span}(S)$, so $f \in \text{Ann}(\text{span}(S))$.

(b) (10 points) Now, let U and W be two subspaces of a V. Prove that $\text{Ann}(U + W) = \text{Ann}(U) \cap \text{Ann}(W)$.

Solution: We'll show both inclusions: First let $f \in \text{Ann}(U + W)$. Then f annihilates any vector of the form $\mathbf{u} + \mathbf{w}$ for $\mathbf{u} \in U$, $\mathbf{w} \in W$. In particular, f annihilates any $\mathbf{u} \in U$ and any $\mathbf{w} \in W$ (which can be represented as $\mathbf{u} + 0$ and $0 + \mathbf{w}$ respectively), so $f \in \text{Ann}(U) \cap \text{Ann}(W)$.

Next, suppose $f \in \text{Ann}(U) \cap \text{Ann}(W)$. Then f annihilates any vector in U and any vector in W. Thus, note that if $\mathbf{u} \in U$ and $\mathbf{w} \in W$, by linearity,

$$f(\mathbf{u} + \mathbf{w}) = f(\mathbf{u}) + f(\mathbf{w}) = 0$$

which means f annihilates any element in $U + W$, i.e. $f \in \text{Ann}(U + W)$.

(c) (5 points) If W is a subspace of V, let V/W denote the quotient of V by W and let $(V/W)^*$ denote its linear dual. Construct a *canonical* (not using a basis) linear map

$$f_W : \text{Ann}(W) \longrightarrow (\text{V/W})^*.$$

other than the zero map (in fact, this map should be invertible). Also, define the inverse f_W^{-1} of this map (though you do not need to carefully check that $f_W \circ f_W^{-1} = I$ and $f_W^{-1} \circ f_W = I$.)

Solution: Let $T \in \text{Ann}(W)$, so T is a linear map $T : V \to \mathbb{F}$ that sends $W \to 0$. By the homework and class, that means T induces a well-defined map on the quotient $\bar{T} : V/W \to \mathbb{F}$, so $\bar{T} \in (V/W)^*$. Define $f_W(T) = \bar{T}$.

Let $\pi_W : V \to V/w$ denote the natural projection map we defined in homework. Then to a linear transformation $S : V/W \to \mathbb{F}$, define

$$f_W^{-1}(S) := S \circ \pi_W : V \to \mathbb{F}.$$

Note that since $\ker \pi_W = W$, $S \circ \pi_W$ annihilates W, so indeed, $S \circ \pi_W \in \text{Ann}(W)$.

Common mistakes:

(a)
- Neglecting to check that $Ann(Span(S)) \subseteq Ann(S)$
- Assuming that S was finite.
- Attempting to consider an infinite sum of vectors, which is undefined.

(b)
- Showing only one of the inclusions and claiming that the other was obvious.

- Attempting to give a vague verbal description of the subspaces involved – "This space is the set of all operators which do this and that to U and V, and so is the other side, so they're equal".

(c) - Defining a function which didn't have the correct domain and/or codomain.

4. (20 points total, 10 points each) *Linear maps, kernels, and images.*

(a) Let $T : V \to W$ and $S : W \to U$ be linear maps between finite-dimensional vector spaces. Prove that if T is surjective, then

$$\dim(\ker ST) = \dim(\ker S) + \dim(\ker T).$$

Solution: We will apply the rank-nullity theorem to each of the maps S, T and ST. We have that T is surjective, so $im(T) = W$ and $im(ST) = im(S)$:

$$\begin{aligned}
\dim(\ker(S)) + \dim(\ker(T)) &= \dim(W) - \dim(im(S)) + \dim(V) - \dim(im(T)) \\
&= \dim(W) - \dim(im(S)) + \dim(V) - \dim(W) \\
&= \dim(V) - \dim(im(S)) \\
&= \dim(V) - \dim(im(ST)) \\
&= \dim(\ker(ST)).
\end{aligned}$$

Therefore $\dim(\ker(S)) + \dim(\ker(T)) = \dim(\ker(ST))$, as required.

(b) Now, suppose suppose $S, T \in \mathcal{L}(V)$, and V is finite-dimensional. Prove that $\ker S \subset \ker T$ if and only if there exists a linear map $R \in \mathcal{L}(V)$ with $T = RS$.

Solution: If there exists a linear map R with $T = RS$, then let v be any element of $\ker(S)$. Then $T(v) = R(S(v)) = R(0) = 0$, so $v \in \ker(T)$. Therefore $\ker(S) \subseteq \ker(T)$.

Now, assume that $\ker(S) \subseteq \ker(T)$. We would like to define a linear map R such that $T(v) = R(S(v))$ for each $v \in V$. Firstly, define R on the image of S by $R(S(v)) = T(v)$. We need to check that this definition is well-defined. That is, if $S(v_1) = S(v_2)$ then $R(S(v_1)) = R(S(v_2))$.

If $S(v_1) = S(v_2)$, then $S(v_1 - v_2) = 0$, so $(v_1 - v_2) \in \ker(S)$, so $(v_1 - v_2) \in \ker(T)$, which means that $T(v_1 - v_2) = 0$ and so $T(v_1) = T(v_2)$. Hence by the definition of R, $R(S(v_1)) = R(S(v_2))$. Therefore our map R is well-defined on its domain, $im(S)$, and satisfies $R(S(v)) = T(v)$ for any $v \in V$.

Finally, extend a basis of $im(S)$ to a basis of V and define R arbitrarily on that basis. This produces a map $R \in \mathcal{L}(V)$ with $T = RS$.

5. (15 points) Let V denote the subspace of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ spanned by the functions $\cos x$, $\sin x$, $x \cos x$, and $x \sin x$. Let $T : V \to V$ be the linear map defined by

$$(Tf)(x) := f(x + \pi) + f(-x).$$

Find an eigenbasis of V with respect to the linear map T. What is the matrix of T with respect to this basis? (Don't forget: $\sin(x+\pi) = -\sin x$, and $\cos(x+\pi) = -\cos x$).

Solution: First, we apply T to each element of the given basis:

$$\begin{aligned} T(\cos(x)) &= 0 \\ T(\sin(x)) &= -2\sin(x) \\ T(x\cos(x)) &= -2x\cos(x) - \pi\cos(x) \\ T(x\sin(x)) &= -\pi\sin(x) \end{aligned}$$

This shows that $\cos(x)$ and $\sin(x)$ are eigenvectors of T with eigenvalues 0 and -2. To find two more eigenvectors, we try linear combinations of $\cos(x)$ and $x\cos(x)$ and of $\sin(x)$ and $x\sin(x)$. (To motivate this, look at what T does to each of these vectors).

We find that $(x\cos(x) - \frac{\pi}{2}\cos(x))$ and $(x\sin(x) - \frac{\pi}{2}\sin(x))$ are eigenvectors of T with eigenvalues -2 and 0. It is easily checked that the four eigenvalues we have given are linearly independent, thus are a basis of V. (Adding a multiple of one vector to another does not change the subspace that they span).

The matrix of T with respect to the eigenbasis

$$\{\cos(x), \sin(x), x\cos(x) - \frac{\pi}{2}\cos(x), x\sin(x) - \frac{\pi}{2}\sin(x)\}$$

is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Common mistakes:

- Finding that $\cos(x)$ and $\sin(x)$ are eigenvectors and the other two are not, and concluding that there are only two eigenvectors.

6. (15 points) *An invariant subspace.* Let V be a vector space with dimension n. Suppose $T : V \to V$ is a linear transformation with n *distinct* eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Set

$$\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_n$$

and consider the subspace W spanned by the vectors $\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, T^3\mathbf{v}, \ldots\}$ (W is the smallest T-invariant subspace containing $\mathbf{v}$). Prove that $W = V$.

Solution: Note that W is a span of elements which are all in V, so $W \subset V$. Thus, in order to show that $W = V$, it suffices to show that $\dim(W) = n$, which we will do by showing that the collection $\{v, Tv, T^2v, \ldots, T^{n-1}v\}$ is linearly independent. Consider any linear dependence relation

$$a_0 v + a_1 Tv + \ldots + a_{n-1} T^{n-1} v = 0. \tag{2}$$

Because each v_j is an eigenvector of T with eigenvalue λ_j, we have that

$$\begin{aligned} T^i(v) &= T^i(v_1 + v_2 + \ldots + v_n) \\ &= \lambda_1^i v_1 + \lambda_2^i v_2 + \ldots + \lambda_n^i v_n. \end{aligned}$$

Substituting this into Equation 2, we get that

$$p(\lambda_1)v_1 + p(\lambda_2)v_2 + \ldots + p(\lambda_n)v_n = 0,$$

where $p(x)$ is the polynomial $a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$. But eigenvectors corresponding to different eigenvalues are linearly independent, so

$$p(\lambda_1) = p(\lambda_2) = \ldots = p(\lambda_n) = 0.$$

Therefore the polynomial p has n distinct roots, $\lambda_1, \ldots, \lambda_n$. But p has degree at most $n-1$, so p must be the zero polynomial. Hence, each a_i is zero.

We have shown that if $a_0 v + a_1 Tv + \ldots + a_{n-1}T^{n-1}v = 0$ then each a_i is zero, so the set $\{v, Tv, T^2 v, \ldots, T^{n-1}v\}$ is linearly independent. Therefore $\dim(W) = n$, so $W = V$, as required.

7. (20 points total) *Matrices and polynomials.*

(a) (5 points) Let V and W be finite-dimensional vector spaces over $\mathbb{F}$, and $T : V \to W$ be a linear map. Give a definition of the matrix of T with respect to a pair of bases (one for V and one for W). **Solution**: Given a basis $\underline{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of V and $\underline{\mathbf{w}} = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ of W, the matrix of T with respect to $\underline{\mathbf{v}}, \underline{\mathbf{w}}$, denoted

$$\mathcal{M}(T, \underline{\mathbf{v}}, \underline{\mathbf{w}})$$

is a particular $m \times n$ array of scalars, indexed as

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \tag{3}$$

where a_{ij} is the scalar such that, when $T\mathbf{v}_j$ is expressed as a linear combination of $\underline{\mathbf{w}}$, a_{ij} is the coefficient of $\mathbf{w}_i$. Namely, the scalars a_{ij} are the unique scalars that satisfy

$$T\mathbf{v}_j = a_{1j}\mathbf{w}_1 + \cdots + a_{ij}\mathbf{w}_i + \cdots + a_{mj}\mathbf{w}_m.$$

(b) (15 points) Let $V = \mathcal{P}_2(\mathbb{R})$ denote the vector space of polynomials of degree ≤ 2 with real coefficients. Consider the standard basis

$$\underline{\mathbf{v}} = (\mathbf{v}_1 = 1, \mathbf{v}_2 = x, \mathbf{v}_3 = x^2)$$

of $\mathcal{P}_2(\mathbb{R})$. On homework, you showed that the dual vector space $V^* = \mathcal{L}(V, \mathbb{R})$ has an induced *dual basis*

$$\underline{\mathbf{v}}^* := (\mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*),$$

where $\mathbf{v}_i^*$ is the linear map from V to $\mathbb{R}$ defined by

$$\mathbf{v}_i^*(\mathbf{v}_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i. \end{cases}$$

Now, let $T : V \to V^*$ be the linear map defined by

$$p \mapsto f_p$$

where $f_p \in V^* = (\mathcal{P}_2(\mathbb{R}))^*$ is the functional defined by

$$f_p(q) := \int_{-1}^{1} p(x)q(x)dx$$

(You do not need to prove this is a linear map). Determine the matrix of T with respect to the bases $\underline{\mathbf{v}}$, $\underline{\mathbf{v}}^*$.

Solution: We need to apply the recipe from part (a). Namely, we need to express $T\mathbf{v}_1$, $T\mathbf{v}_2$ and $T\mathbf{v}_3$ as linear combinations of elements of $\underline{\mathbf{v}}^*$.

First, $T\mathbf{v}_1 = T(1) = f_1$. f_1 is the functional whose values on basis elements $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are

$$f_1(1) = \int_{-1}^{1} 1dx = x|_{-1}^{1} = 2$$
$$f_1(x) = \int_{-1}^{1} xdx = \frac{1}{2}x^2|_{-1}^{1} = 0$$
$$f_1(x^2) = \int_{-1}^{1} x^2dx = \frac{1}{3}x^3|_{-1}^{1} = \frac{2}{3}.$$

Since $2\mathbf{v}_1^* + \frac{2}{3}\mathbf{v}_3^*$ is a functional taking the same values on the basis elements $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we conclude that

$$T\mathbf{v}_1 = f_1 = 2\mathbf{v}_1^* + \frac{2}{3}\mathbf{v}_3^*.$$

Now, $T\mathbf{v}_2 = T(x) = f_x$, which is the functional whose values on the basis $\underline{\mathbf{v}}$ are

$$f_x(1) = \int_{-1}^{1} x dx = 0.$$
$$f_x(x) = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$
$$f_x(x^2) = \int_{-1}^{1} x^3 dx = \frac{1}{4}x^4|_{-1}^{1} = 0.$$

(where some of these integrals agree with earlier calculations). Hence, in a similar fashion $T\mathbf{v}_2 = f_x = \frac{2}{3}\mathbf{v}_2^*$.

Finally, $T\mathbf{v}_3 = T(x^2) = f_{x^2}$, which is the functional whose values on the basis $\underline{\mathbf{v}}$ are

$$f_{x^2}(1) = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$
$$f_{x^2}(x) = \int_{-1}^{1} x^3 dx = \frac{1}{4}x^4|_{-1}^{1} = 0$$
$$f_{x^2}(x^2) = \int_{-1}^{1} x^4 dx = \frac{1}{5}x^5|_{-1}^{1} = \frac{2}{5}.$$

i.e. $T\mathbf{v}_3 = f_{x^2} = \frac{2}{3}\mathbf{v}_1^* + \frac{2}{5}\mathbf{v}_3^*$.

Applying this to the defintion of matrix, we finally get that

$$\mathcal{M}(T, \underline{\mathbf{v}}, \underline{\mathbf{v}}^*) = \begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5}. \end{bmatrix}. \tag{4}$$

Common mistakes:

- Mixing up the rows and columns of the matrix, or the corresponding subscripts.
- Misunderstanding the dual space $(\mathcal{P}_2(\mathbb{R}))^*$. For example, only attempting to apply f_1 to 1, f_x to x and f_{x^2} to x^2.