Math 113 Supplementary Notes: Understanding the Cross Product

If you have taken a computational linear algebra course, you may have learned of a special vector-product on $\mathbb{R}^3$, the *cross product*, denoted $\times$. This product is defined by an explicit formula

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3) = (x_2y_3 - y_3x_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

and satisfies some crucial properties:

- If $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, then their cross product is zero,
- The map $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \times \mathbf{w}$ is bilinear,
- $\mathbf{v} \times \mathbf{w}$ is always orthogonal to $\mathbf{v}$ and $\mathbf{w}$.
- $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$, so the cross product is in fact an *alternating bilinear* product,
- The Euclidean norm $||\mathbf{v} \times \mathbf{w}||$ is $||\mathbf{v}||||\mathbf{w}|| \sin \theta$, where θ is the "angle between $\mathbf{v}$ and $\mathbf{w}$". In other words, this norm is the Euclidean area of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$.
- The direction that $\mathbf{v} \times \mathbf{w}$ points in along the line orthogonal to $\text{span}(\mathbf{v}, \mathbf{w})$ can be calculated using a "right hand rule."

At first glance, the cross product is mysterious—and it is not at all clear why it should only exist on $\mathbb{R}^3$ (and other 3-dimensional vector spaces, it will turn out), but not in higher dimension! On the other hand, the fact that cross products are alternating bilinear should suggest a relationship to the wedge product.

Let's now explain. Let V be a vector space of dimension 3, and $\mathbf{v}$, $\mathbf{w}$ a pair of vectors. We can then take the wedge product

$$\mathbf{v} \wedge \mathbf{w} \in \bigwedge{}^2 V,$$

Now let's notice that

$$\dim(\bigwedge{}^2 V) = \binom{3}{2} = 3 = \dim(V);$$

which crucially uses the fact that V is 3-dimensional (otherwise the dimension of $\bigwedge^2 V$ is much bigger than that of V!). Thus, if we had an isomorphism

$$\star : \bigwedge{}^2 \xrightarrow{\sim} V$$

we could compose this map with the exterior product

$$\psi : V \times V \longrightarrow \bigwedge{}^2 V$$

to get an alternating bilinear product

$$\star \circ \psi : V \times V \longrightarrow V.$$

Let's describe how we choose the isomorphism $\star$; this requires some additional choices which have implicitly been made for the case of $\mathbb{R}^3$.

First, we need to pick an isomorphism, which we call a **signed volume**

$$vol : \bigwedge{}^3 V \xrightarrow{\sim} \mathbb{F}.$$

Since $\dim \bigwedge^3 V = 1$, vol is determined entirely by which vector in $\bigwedge^3 V$ is sent to 1. This vector is called a **volume form**. For $\mathbb{R}^3$ for example, the standard choice of volume form is the 3-blade

$$e_1 \wedge e_2 \wedge e_3$$

Next, the signed volume gives us a map

$$\bigwedge^2 V \longrightarrow V^*$$

by

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \longmapsto F_{\mathbf{v}_1,\mathbf{v}_2}$$

where $F_{\mathbf{v}_1,\mathbf{v}_2} \in V^*$ is the functional

$$F_{\mathbf{v}_1,\mathbf{v}_2}(\mathbf{w}) = vol(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{w}).$$

You can check on $\mathbb{R}^3$ that with the volume form specified above, this basis of 2-blades is sent to the following set of functionals.

$$\begin{aligned} e_1 \wedge e_2 &\mapsto \langle e_3, \cdot \rangle \\ e_2 \wedge e_3 &\mapsto \langle e_1, \cdot \rangle \\ e_1 \wedge e_3 &\mapsto \langle -e_2, \cdot \rangle. \end{aligned}$$

Finally, if one has an inner product on V (as we do in $\mathbb{R}^3$), one can identify the functional $F_{\mathbf{v}_1,\mathbf{v}_2}$ as $\langle \mathbf{u}_{\mathbf{v}_1,\mathbf{v}_2}, \cdot \rangle$ for a unique vector $\mathbf{u}_{\mathbf{v}_1,\mathbf{v}_2}$. (In other words, we're using the inner product to give us an isomorphism from V^* to V). Thus, using a **signed volume** and an **inner product**, we have defined a map

$$\begin{aligned} \star : \bigwedge^2 V &\longrightarrow V \\ \mathbf{v}_1 \wedge \mathbf{v}_2 &\longmapsto \mathbf{u}_{\mathbf{v}_1,\mathbf{v}_2} \end{aligned}$$

as the composition

$$\bigwedge^2 V \longrightarrow V^* \longrightarrow V$$

where the first map used the volume, and the second map used the inner product.

On $\mathbb{R}^3$, this map behaves as follows:

$$\begin{aligned} e_1 \wedge e_2 &\mapsto e_3 \\ e_2 \wedge e_3 &\mapsto e_1 \\ e_1 \wedge e_3 &\mapsto -e_2 \end{aligned}$$

The composed map

$$\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\psi} \bigwedge^2 \mathbb{R}^3 \xrightarrow{\star} \mathbb{R}^3$$

is exactly the **cross product**, as it is alternating bilinear and satisfies

$$(e_1, e_2) \mapsto e_3$$
$$(e_2, e_3) \mapsto e_1$$
$$(e_1, e_3) \mapsto -e_2$$

REMARK 1. *$\star$ is part of a more general series of maps*

$$\star : \bigwedge^k V \longrightarrow \bigwedge^{n-k} V$$

where $n = \dim V$, which are defined using signed volumes and inner products on $\bigwedge^{n-k} V$ (induced by inner products on V). Collectively, these are called **Hodge star operators**. *With respect to inner products on $\bigwedge^k V$ and $\bigwedge^{n-k} V$, each such map $\star$ is an isometry.*