Math 113: Tensor Products

1. Fundamental properties

This past week, you proved some first properties of the tensor product $V \otimes W$ of a pair of vector spaces V and W. This week, I want to rehash some fundamental properties of the tensor product, that you you are welcome to take as a working definition from here forwards.

Let V and W be two finite-dimensional vector spaces over $\mathbb{F}$. The tensor product $V \otimes W$ is a vector space over $\mathbb{F}$, equipped with a map

$$\phi : V \times W \to V \otimes W.$$

We often refer to the vector $\phi(\mathbf{v}, \mathbf{w})$ in $V \otimes W$ as the vector $\mathbf{v} \otimes \mathbf{w}$. The vector space $V \otimes W$ and the map ϕ satisfy the following fundamental properties:

1. The map ϕ is *bilinear*. (Why should it be bilinear? Recall that ordinary multiplication of real numbers $(x, y) \mapsto x \cdot y$ is a bilinear map, so this is one generalization of multiplying). We sometimes call ϕ *vector multiplication*. Call any vector in the image of ϕ (any element in $V \otimes W$ of the form $\mathbf{v} \otimes \mathbf{w}$) a **pure tensor**.

   By bilinearity of ϕ, we see that

   $$(a\mathbf{v} + b\mathbf{v}') \otimes \mathbf{w} = \phi(a\mathbf{v} + b\mathbf{v}', \mathbf{w}) = a\phi(\mathbf{v}, \mathbf{w}) + b\phi(\mathbf{v}', \mathbf{w}) = a\mathbf{v} \otimes \mathbf{w} + b\mathbf{v}' \otimes \mathbf{w},$$

   and similarly

   $$\mathbf{v} \otimes (a\mathbf{w} + b\mathbf{w}') = a\mathbf{v} \otimes \mathbf{w} + b\mathbf{v} \otimes \mathbf{w}'.$$

2. If $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is any basis of V and $(\mathbf{w}_1, \dots, \mathbf{w}_m)$ is any basis of W, then the collection

   $$\{\mathbf{v}_i \otimes \mathbf{w}_j = \phi(\mathbf{v}_i, \mathbf{w}_j),\ 1 \le i \le n, 1 \le j \le m\}$$

   is always a basis of $V \otimes W$. Thus,

   $$\dim(V \otimes W) = \dim V \cdot \dim W. \tag{1}$$

3. Not every element in $V \otimes W$ is necessarily of the form $\mathbf{v} \otimes \mathbf{w}$. One way to see this is to work in terms of a basis: note that given any pair $(\mathbf{v}, \mathbf{w})$, we can write them in terms of a basis as $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ and $\mathbf{w} = b_1\mathbf{w}_1 + \cdots + b_m\mathbf{w}_m$, for some scalars $a_1, \dots, a_n$, $b_1, \dots, b_m$. This implies that

   $$\phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \otimes \mathbf{w} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \cdot b_j \mathbf{v}_i \otimes \mathbf{w}_j. \tag{2}$$

But a general element in $V \otimes W$ is of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \mathbf{v}_i \otimes \mathbf{w}_j,$$

where c_{ij} is a collection of mn scalars. Such a collection cannot always be written as $a_i \cdot b_j$ for a collection of $m+n$ scalars $a_1, \ldots, a_n, b_1, \ldots, b_m$. This is equivalent to the fact that not every polynomial in two variables x, y can be written as the product $(a_0+a_1x+\cdots+a_nx^n)\cdot(b_0+b_1y+\cdots+b_ny^m)$. For example, $xy+1$ cannot be written this way.

We conclude that while every element in $V \otimes W$ is a sum of elements of the form $\mathbf{v} \otimes \mathbf{w}$, not every element is necessarily of this form (unless the dimension of either V or W is 0 or 1).

4. On the past homework, you proved, or attempted to prove, the following proposition:

   PROPOSITION 1.1. *If $F : V \times W \to X$ is any bilinear map, then F can be written uniquely as a composition*

   $$V \times W \xrightarrow{\phi} V \otimes W \xrightarrow{\underline{F}} X,$$

   where ϕ is the multiplication map, and $\underline{F}$ is a linear map.

   Let's first note we already know what $\underline{F} : V \otimes W$ has to be on pure tensors:

   $$\underline{F}(\mathbf{v} \otimes \mathbf{w}) = \underline{F}(\phi(\mathbf{v}, \mathbf{w})) = F(\mathbf{v}, \mathbf{w})$$

   Since a general vector in $V \otimes W$ is a sum of some pure tensors, and we want $\underline{F}$ to be linear, it is determined by its value on pure tensors (extending to all of $V \otimes W$ by linearity). The proposition guarantees that the result $\underline{F}$ is indeed a linear map.

   COROLLARY 1.1. *Let $\mathcal{L}(V \times W, X)$ denote the space of bilinear maps from $V \times W$ into X. Then, there is an isomorphism*

   $$\mathcal{L}(V \times W, X) \cong \mathcal{L}(V \otimes W, X)$$

   where the right hand side denotes the space of linear maps out of the tensor product.

   The above proposition has important ramifications, some of which you explored on HW 7. Crucially for us, it will give us a way of constructing linear maps out of the tensor product.

Constructing linear maps $T : V \otimes W \to X$: The above discussion gives us two ways of constructing linear maps out of the tensor product of vector spaces:

- Just define the values of the map on a basis $\{\mathbf{v}_i \otimes \mathbf{w}_j\}$, and extend by linearity; or

- Define the T on pure tensors, of the form $\mathbf{v} \otimes \mathbf{w}$, and extend by linearity to all of $V \otimes W$. By part 4 above, *as long as the map*

  $$\begin{aligned} \hat{T} : V \times W &\to X \\ (\mathbf{v}, \mathbf{w}) &\mapsto T(\mathbf{v} \otimes \mathbf{w}) \end{aligned}$$

  is bilinear, you are guaranteed that T is linear!.

2. First examples, and applications

Given the intimate relationship of tensor products to bilinear maps (property 4), it should not be surprising that tensor products arise whenever there are bilinear maps lurking around. In this section, we'll see a few examples of where tensor products might begin to arise, and examples of tensor products.

(1) $\mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{mn}$. It might be fairer to say that $\mathbb{R}^m \otimes \mathbb{R}^n$ is isomorphic to the space $\mathrm{Mat}(m, n, \mathbb{R})$ of $m \times n$ real matrices, which also has dimension mn. The correspondence is straightforward to describe: $\mathbb{R}^m \otimes \mathbb{R}^n$ has a basis coming from the pairs of standard basis vectors:

$$\{e_i \otimes e_j,\ 1 \leq i \leq m,\ 1 \leq j \leq n,$$

and $\mathrm{Mat}(m, n, \mathbb{R})$ has the standard basis

$$e_{ij},\ 1 \leq i \leq m,\ 1 \leq j \leq n$$

where e_{ij} is the matrix that has a 1 in component (i, j) and 0 elsewhere. One choice of isomorphism is the identification

$$e_i \otimes e_j \mapsto e_{ij}.$$

(2) Let $\mathbb{F}[x]_n$ denote the vector space of polynomials in x with $\mathbb{F}$ coefficients, and degree $\leq n$. Let $\mathbb{F}[y]_m$ denote the vector space of polynomials in y with $\mathbb{F}$ coefficients and degree $\leq m$. And let $\mathbb{F}[x, y]_{n,m}$ denote the vector space of polynomials in two variables x, y with $\mathbb{F}$ coefficients, such that the degree of x and the degree of y in any term are $\leq n$ and m respectively. Then, there is an isomorphism

$$\mathbb{F}[x]_n \otimes \mathbb{F}[y]_m \to \mathbb{F}[x, y]_{n,m}$$

given by identifying the basis element

$$x^i \otimes y^j$$

of the left hand side with the basis element

$$x^i y^j$$

on the right hand side.

(3) *Inner products*. An inner product structure on a real vector space V is a bilinear map

$$F = \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

satisfying some additional conditions (positivity, defininiteness, symmetry).

By 4 in the previous section, an inner product structure induces a *linear* map

$$\underline{F} : V \otimes V \to \mathbb{R},$$

i.e., an element

$$\underline{F} \in (V \otimes V)^*.$$

Positivity, definitiness, and symmetry then translate into properties of $\underline{F}$. This is occasionally a useful perspective.

(4) *Relation to spaces of linear maps.* Given that $V \otimes W$ has the same dimension as $\mathcal{L}(V, W)$, one might expect some sort of relationship. There is not a canonical one, but there is a canonical relationship between $V^* \otimes W$ and $\mathcal{L}(V, W)$! (V^* has the same dimension as V, as we know, so these two spaces have the same dimension too). On homework this week, you will construct a canonical linear map

$$V^* \otimes W \longrightarrow \mathcal{L}(V, W)$$

(To do this: recall the advice given in the previous section about constructing linear maps out of the tensor product). This map is an isomorphism whenever V and W are finite dimensional.

We'll use this correspondence on Homework 8 to give a basis-free definition of the *trace* of a linear map.