

Wedge Products and the Determinant

Math 113

1. Goals and Motivations

The goal of these lecture notes are the following:

- To give a basis-free definition of the determinant $\det(T)$ of a linear map $T : V \rightarrow V$, using *wedge products*.
- Define the characteristic polynomial of a linear operator T in a way that can be calculated using any basis of V , not just one for which the matrix of T is upper triangular. (The characteristic polynomial will be defined as $\det(zI - T)$, and its properties will follow from those of the determinant).
- Understand something about the relationship of determinants and wedge products to volumes in vector spaces.

We begin with some geometric intuition below before proceeding into the algebraic definitions. For this lecture, we assume some (but not much) familiarity with tensor products, following earlier Math 113 lecture notes.

2. (Signed) lengths and volumes

Let V be a vector space. A natural geometric notion in V which we have been studying is that of the *length* of vectors in V . Using the notion of an *inner product* on V , we defined the length of a vector \mathbf{v} as the norm $\|\mathbf{v}\|$, and then studied various questions regarding length.

Prior to introducing inner products, we observed that V already “knows” some preliminary information about lengths:

- For any definition of length, the zero vector $\mathbf{0}$ should have length 0.
- If a vector \mathbf{w} is in the span of another vector \mathbf{v} , so $\mathbf{w} = c\mathbf{v}$, for some c , then the length of \mathbf{w} should be $|c|$ times the length of \mathbf{v} .

The presence of an absolute value in the second bullet point is somewhat unsatisfying from the point of linear algebra, as it means length is not a linear function. We can restore this linearity by first discussing signed lengths, and then taking absolute values at the end.

DEFINITION 2.1. *Let V be a 1-dimensional vector space. A **signed length** is a linear map $l : V \rightarrow \mathbb{F}$. Given a signed length l , the **unsigned length** of a vector \mathbf{v} is the absolute value $|l(\mathbf{v})|$.*

Let us try to generalize this story to understanding volumes. First a warm-up: consider areas in a 2-dimensional vector space:

DEFINITION 2.2. *Given a pair of vectors \mathbf{v} and \mathbf{w} in a 2-dimensional vector space V , the **parallelogram** of \mathbf{v} and \mathbf{w} is the region*

$$p(\mathbf{v}, \mathbf{w}) := \{a\mathbf{v} + b\mathbf{w} \mid 0 \leq a, b \leq 1\}.$$

More generally, the k -dimensional parallelipiped associated to a collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the set

$$p(\mathbf{v}_1, \dots, \mathbf{v}_k) := \{a_1\mathbf{v}_1 + \dots + a_k\mathbf{v}_k \mid 0 \leq a_i \leq 1\}$$

We would like to understand notions of *area* and *relative area*. Just like length, an area function is an additional structure one assigns to a vector space. Let's see what properties such a function should have, using standard area in \mathbb{R}^2 as an example. In \mathbb{R}^2 , the standard unsigned area of the parallelogram is

$$\|\mathbf{v}\| \|\mathbf{w}\| \sin \theta,$$

where θ is the angle between \mathbf{v} and \mathbf{w} . One can verify geometrically that this area satisfies the following properties:

1. The area of parallelograms with one edge scaled by c is $|c|$ times the area of the original parallelogram, i.e. $p(c\mathbf{v}, \mathbf{w}) = p(\mathbf{v}, c\mathbf{w}) = |c|p(\mathbf{v}, \mathbf{w})$.
2. The area of $p(\mathbf{v}, \mathbf{v})$ equals 0 for any \mathbf{v} . More generally, the area of $p(\mathbf{v}, \mathbf{w})$ equals 0 if \mathbf{v} and \mathbf{w} are parallel. (this condition is called the **alternating** condition).
3. The area of $p(\mathbf{v} + \mathbf{v}', \mathbf{w})$ equals \pm the area of $p(\mathbf{v}, \mathbf{w}) \pm$ the area of $p(\mathbf{v}', \mathbf{w})$. Whether the signs of each of these terms is positive or negative depends on the relative directions of \mathbf{v} , \mathbf{v}' , and \mathbf{w} .

The absolute value in 1 and the \pm signs in 3 prevent this area function from being *bilinear*, even though it seems be close. As with lengths, we'll repair this by considering *signed areas* instead. For example, in \mathbb{R}^2 , we'll add a \pm sign to the area defined above (the sign will be $+$ if \mathbf{w} is ≤ 180 degrees from \mathbf{v} in the counterclockwise direction and $-$ if \mathbf{v} is ≤ 180 degrees from \mathbf{w} in the counterclockwise direction). The resulting signed area function is the first example of an *alternating, bilinear map*.

3. Alternating multilinear maps

We would like to generalize the above story to volumes in higher dimensions. Thus, we first make the following definitions:

DEFINITION 3.1. Let V and X be vector spaces. A map

$$F : \underbrace{V \times \dots \times V}_{k \text{ copies}} \longrightarrow X$$

is **multilinear** or **k -linear** if it is linear in each slot, that is if

$$F(\mathbf{v}_1, \dots, a\mathbf{v}_i + b\mathbf{v}'_i, \dots, \mathbf{v}_k) = aF(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_k) + bF(\mathbf{v}_1, \dots, \mathbf{v}'_i, \dots, \mathbf{v}_k).$$

A map

$$F : \underbrace{V \times \dots \times V}_k \longrightarrow X$$

is **alternating** if, whenever two of the inputs to F are the same vector, the output is $\mathbf{0}$. That is, if

$$F(\cdots, \mathbf{v}, \cdots, \mathbf{v}, \cdots) = 0 \text{ for all } \mathbf{v}.$$

PROPOSITION 3.1. Suppose $F : \underbrace{V \times \cdots \times V}_k \rightarrow X$ is multilinear. Then it is alternating if and only if swapping two inputs flips the sign of F , i.e. if

$$(1) \quad \begin{aligned} F(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{j-1}, \mathbf{w}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k) \\ = -F(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{w}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{j-1}, \mathbf{v}, \dots, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k). \end{aligned}$$

for all \mathbf{v}, \mathbf{w} .

PROOF. Suppose F is multilinear and alternating. Then, note that for any \mathbf{v}, \mathbf{w} :

$$\begin{aligned} 0 &= F(\dots, (\mathbf{v} + \mathbf{w}), \dots, (\mathbf{v} + \mathbf{w}), \dots) \\ &= F(\dots, \mathbf{v}, \dots, \mathbf{v}, \dots) + F(\dots, \mathbf{w}, \dots, \mathbf{w}, \dots) \\ &\quad + F(\dots, \mathbf{v}, \dots, \mathbf{w}, \dots) + F(\dots, \mathbf{w}, \dots, \mathbf{v}, \dots) \\ &= F(\dots, \mathbf{v}, \dots, \mathbf{w}, \dots) + F(\dots, \mathbf{w}, \dots, \mathbf{v}, \dots), \end{aligned}$$

where we have used the multilinearity and alternating conditions. This implies one direction.

In the other direction, if (1) holds for all \mathbf{v}, \mathbf{w} then, looking at the case $\mathbf{v} = \mathbf{w}$, we see that

$$\begin{aligned} F(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{j-1}, \mathbf{v}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k) \\ = -F(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{j-1}, \mathbf{v}, \dots, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k), \end{aligned}$$

which is only possible if both expressions equal 0. □

We will denote by

$$\mathcal{L}(\underbrace{V \times \cdots \times V}_k, X)$$

the vector space of k -linear maps and by

$$\mathcal{L}^{Alt}(\underbrace{V \times \cdots \times V}_k, X)$$

the vector subspace of alternating k -linear maps.

EXERCISE 3.1. Prove that these spaces are indeed vector spaces.

DEFINITION 3.2. A (signed) k -volume on a k -dimensional vector space is an **alternating, multilinear map**

$$\underbrace{V \times \cdots \times V}_k \rightarrow \mathbb{F}$$

An unsigned k -volume is the absolute value of some signed volume.

The usual notion of Euclidean volume arises as the absolute value of a signed volume, for instance.

4. Wedge products: a working definition

Wedge products arise in a similar manner that tensor products do. Namely, wedge products provide spaces \underline{V} (depending on k) such that alternating k -linear maps from V to X are the same as linear maps from \underline{V} to X .

The k th exterior (or wedge) product

$$\bigwedge^k V$$

is a vector space, equipped with a map

$$(2) \quad \psi : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \longrightarrow \bigwedge^k V,$$

called *exterior multiplication*. We commonly refer to the element $\psi(\mathbf{v}_1, \dots, \mathbf{v}_k)$ as a **pure/simple k -wedge** (or **k -blade**) and denote it by $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$.

The pair $(\bigwedge^k V, \psi)$ satisfy the following key properties:

1. ψ is *alternating multilinear*, namely

$$(3) \quad \mathbf{v}_1 \wedge \cdots \wedge (a\mathbf{v}_i + b\mathbf{v}'_i) \wedge \cdots \wedge \mathbf{v}_k = a\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_i \wedge \cdots \wedge \mathbf{v}_k + b\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}'_i \wedge \cdots \wedge \mathbf{v}_k.$$

$$(4) \quad \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v} \wedge \cdots \wedge \mathbf{v} \wedge \cdots \wedge \mathbf{v}_k = 0$$

As before, these two conditions imply that

$$(5) \quad \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_i \wedge \cdots \wedge \mathbf{v}_j \wedge \cdots \wedge \mathbf{v}_k = -\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j \wedge \cdots \wedge \mathbf{v}_i \wedge \cdots \wedge \mathbf{v}_k.$$

2. Given a basis $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ of V , the collection

$$(6) \quad \mathcal{B}_k = \{\mathbf{v}_{i_1} \wedge \cdots \wedge \mathbf{v}_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

forms a basis for $\bigwedge^k V$. The number of elements in \mathcal{B}_k is therefore the number of *unordered collections of size k* in the list $(\mathbf{v}_1, \dots, \mathbf{v}_n)$. (Why?) Thus,

$$(7) \quad \dim \bigwedge^k V = \binom{n}{k}.$$

This is in stark contrast to the k -fold tensor product $V \otimes \cdots \otimes V$, for which the following collection of n^k vectors is a basis: $\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_k} \mid 1 \leq i_j \leq n\}$. To see why such a collection is redundant in $\bigwedge^k V$, let us look at an example:

EXAMPLE 4.1. Consider $\bigwedge^2 V$ for a 2-dimensional vector space V . If $(\mathbf{v}_1, \mathbf{v}_2)$ is a basis of V , note that the collection

$$\mathbf{v}_1 \wedge \mathbf{v}_1, \mathbf{v}_1 \wedge \mathbf{v}_2, \mathbf{v}_2 \wedge \mathbf{v}_1, \mathbf{v}_2 \wedge \mathbf{v}_2$$

is not linearly independent. Namely, $\mathbf{v}_1 \wedge \mathbf{v}_1 = \mathbf{v}_2 \wedge \mathbf{v}_2 = 0$, and $\mathbf{v}_2 \wedge \mathbf{v}_1 = -\mathbf{v}_1 \wedge \mathbf{v}_2$, using the alternating property of wedges.

Thus, the span of this collection is contained in the span of $\mathbf{v}_1 \wedge \mathbf{v}_2$.

Similarly, in general, the collection of all k -fold wedges of basis vectors cannot be linearly independent, as those k -wedges with a repeated basis vector are 0, and those without repeats are equal to ± 1 (depending on the number of vectors one needs to swap) times a vector in \mathcal{B}_k . For instance, if $(\mathbf{v}_1, \dots, \mathbf{v}_4)$ is a basis of V , then note that in $\bigwedge^3 V$,

$$\mathbf{v}_3 \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 = -\mathbf{v}_1 \wedge \mathbf{v}_3 \wedge \mathbf{v}_2 = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3,$$

where we performed two swaps (with the accompanying $-$ signs).

3. Elements in $\bigwedge^k V$ are not necessarily all pure k -wedges; they may instead be a sum of pure k -wedges.

4. If

$$F : \underbrace{V \times \dots \times V}_k \longrightarrow X$$

is any alternating multilinear map, then F factors uniquely as a composition

$$\underbrace{V \times \dots \times V}_k \xrightarrow{\psi} \bigwedge^k V \xrightarrow{\underline{F}} X$$

where ψ is exterior multiplication and \underline{F} is a linear map.

We know what $\underline{F} : \bigwedge^k V \longrightarrow X$ has to be on pure k -wedges:

$$\underline{F}(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k) = \underline{F}(\psi(\mathbf{v}_1, \dots, \mathbf{v}_k)) = F(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

Since a general vector in $\bigwedge^k V$ is a sum of pure wedges, and we want \underline{F} to be linear, \underline{F} is determined by its value on pure wedges, which is in turn determined by F . This property tells us that the resulting \underline{F} is a linear map whenever F is alternating multilinear. (This gives us a way of constructing linear maps out of the wedge product). It also tells us that, as desired,

$$\mathcal{L}^{Alt}(\underbrace{V \times \dots \times V}_k, X) \cong \mathcal{L}(\bigwedge^k V, X).$$

These properties have the following consequences:

PROPOSITION 4.1 (Top exterior power). *Let V be a vector space of dimension n . Then, for $k > n$,*

$$\bigwedge^k V = \{0\},$$

and for $k = n$,

$$\dim \bigwedge^n V = 1$$

with basis $(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_n)$ for any basis $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ of V .

PROPOSITION 4.2 (Wedge dependence lemma). *The k -blade $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k$ equals 0 if and only if the collection $\mathbf{v}_1, \dots, \mathbf{v}_k$ is linearly dependent.*

PROOF. First, suppose $\mathbf{v}_1, \dots, \mathbf{v}_k$ is linearly dependent. Without loss of generality, assume that \mathbf{v}_1 is in the span of $\mathbf{v}_2, \dots, \mathbf{v}_k$, so

$$\mathbf{v}_1 = a_2 \mathbf{v}_2 + \dots + a_k \mathbf{v}_k$$

for some a_2, \dots, a_k . Then

$$\begin{aligned} \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k &= (a_2 \mathbf{v}_2 + \dots + a_k \mathbf{v}_k) \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_k \\ &= \sum_{i=2}^k a_i \mathbf{v}_i \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_i \wedge \dots \wedge \mathbf{v}_k = 0 \end{aligned}$$

where the last equality is because each term in the sum of wedges has a repeated vector.

Going the other direction, suppose that $\mathbf{v}_1, \dots, \mathbf{v}_k$ is linearly independent. Then, it can be extended to a basis $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ of V , so by property 2, the collection of wedges of the form

$$\mathbf{v}_{i_1} \wedge \dots \wedge \mathbf{v}_{i_k}$$

for $i_1 < i_2 < \dots < i_k$ is a basis for $\bigwedge^k V$. But $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k$ is an element of this basis, so it cannot be zero. \square

Similarly,

PROPOSITION 4.3. *Let $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k$ and $\mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_k$ be two non-zero pure k wedges. Then,*

$$\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k = c \mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_k$$

for some non-zero scalar c if and only if

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{span}(\mathbf{w}_1, \dots, \mathbf{w}_k)$$

in V .

PROOF. Let U denote the span of $(\mathbf{w}_1, \dots, \mathbf{w}_k)$ and U' the span of $(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

First, suppose that $U = U'$. Then every vector \mathbf{v}_i can be expressed as a linear combination of the list $(\mathbf{w}_1, \dots, \mathbf{w}_k)$. We can plug these expressions into the simple wedge $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k$ to express it, after applying multilinearity and the alternating condition a number of times, as $c \mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_k$ for some scalar c . c is non-zero precisely because we know both simple wedges $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k$ and $\mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_k$ must be non-zero.

Now, suppose $U \neq U'$, so the intersection $U \cap U'$ has some dimension $l < k$. We can find new collections $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ and $(\mathbf{w}_1, \dots, \mathbf{w}_k)$ so that spans are still U and U' (hence the wedges of these new collections are constant multiples of wedges of the old collections), such that the first l elements of both lists are $\mathbf{u}_1, \dots, \mathbf{u}_l$, a basis for $U \cap U'$. So the first collection is of the form $(\mathbf{u}_1, \dots, \mathbf{u}_l, \mathbf{v}_1, \dots, \mathbf{v}_{k-l})$ and the second collection is of the form $(\mathbf{u}_1, \dots, \mathbf{u}_l, \mathbf{w}_1, \dots, \mathbf{w}_{k-l})$. But now, the larger list

$$(\mathbf{u}_1, \dots, \mathbf{u}_l, \mathbf{v}_1, \dots, \mathbf{v}_{k-l}, \mathbf{w}_1, \dots, \mathbf{w}_{k-l})$$

is linearly independent, so it can be extended to a basis of all of V , which in particular implies by property 2 that the vectors

$$\mathbf{u}_1 \wedge \dots \wedge \mathbf{u}_l \wedge \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{k-l}$$

and

$$\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_l \wedge \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_{k-l}$$

are part of the same basis for $\bigwedge^k V$, so cannot differ by a constant multiple. \square

4.1. An analogy. The above propositions should suggest an analogy that is useful to keep in mind: *vectors in V are to lengths in V as pure k -wedges in $\bigwedge^k V$ are to volumes of parallelpipeds in V .* The meaning is this: there are many possible signed length functions on V , but all of them satisfy

$$\text{length}(\mathbf{w}) = c \cdot \text{length}(\mathbf{v}) \text{ if } \mathbf{w} = c\mathbf{v}.$$

Similarly, there are many possible signed volume functions on V , but all of them satisfy

$$\text{vol}(p(\mathbf{w}_1, \dots, \mathbf{w}_k)) = c \cdot \text{vol}(p(\mathbf{v}_1, \dots, \mathbf{v}_k))$$

if

$$\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k = c\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k.$$

This in turn holds if the span of $\mathbf{w}_1, \dots, \mathbf{w}_k$ is the same as the span of $\mathbf{v}_1, \dots, \mathbf{v}_k$. In that case, the parallelpipeds $p(\mathbf{v}_1, \dots, \mathbf{v}_k)$ and $p(\mathbf{w}_1, \dots, \mathbf{w}_k)$ have the same comparable *relative volume* for any signed volume function: the scalar c .

5. Constructing the exterior product

There is a formal construction of $\bigwedge^k V$, analogous to HW 7, which one can directly use to verify that $\bigwedge^k V$ exists and satisfies properties 1-4. Strictly speaking, one first proves properties 1 and 4, which in turn can be used to prove property 2 (property 3 is automatic from the construction). For the purposes of our class, it would be better to take for granted the existence of $\bigwedge^k V$, use properties 1-4 as its defining properties, and skip this section.

However, for the sake of completeness: one way of defining $\bigwedge^k V$, as in HW7 is as the set of formal sums

$$\sum_{j=1}^k a_j \mathbf{v}_1^j \wedge \cdots \wedge \mathbf{v}_k^j$$

where two such sums are said to be the same if they differ by the application of any relations of the form

$$(8) \quad \mathbf{v}_1 \wedge \cdots \wedge (a\mathbf{v}_i + b\mathbf{v}'_i) \wedge \cdots \wedge \mathbf{v}_k = a\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_i \wedge \cdots \wedge \mathbf{v}_k + b\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}'_i \wedge \cdots \wedge \mathbf{v}_k.$$

$$(9) \quad \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v} \wedge \cdots \wedge \mathbf{v} \wedge \cdots \wedge \mathbf{v}_k = 0$$

$$(10) \quad \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_i \wedge \cdots \wedge \mathbf{v}_j \wedge \cdots \wedge \mathbf{v}_k = -\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j \wedge \cdots \wedge \mathbf{v}_i \wedge \cdots \wedge \mathbf{v}_k.$$

Then, one can define ψ to be the map that takes $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ to the formal sum $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$.

We will not pursue this formal construction further.

6. Induced Maps

Given a linear map

$$T : V \longrightarrow W$$

for any k there is an induced map

$$T_* : \bigwedge^k V \longrightarrow \bigwedge^k W$$

given on pure k -wedges by

$$T_*(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k) = T(\mathbf{v}_1) \wedge \cdots \wedge T(\mathbf{v}_k).$$

PROPOSITION 6.1. T_* is a linear map.

PROOF. We'll not carefully prove this here, but by properties above, this reduces to proving that

$$\begin{aligned} \hat{T} : \underbrace{V \times \cdots \times V}_k &\longrightarrow \bigwedge^k W \\ (\mathbf{v}_1, \dots, \mathbf{v}_k) &\longmapsto (T\mathbf{v}_1) \wedge \cdots \wedge (T\mathbf{v}_k) \end{aligned}$$

is *alternating multilinear*. This in turn follows very quickly from the linearity of T and alternating multilinearity of exterior multiplication in W . \square

Now, let's suppose there is another linear map

$$S : W \longrightarrow U.$$

This induces a map

$$S_* : \bigwedge^k W \longrightarrow \bigwedge^k U$$

so in particular, we can compose S_* with T_* . The next proposition tells us what the composition of these induced maps is in terms of the composition of maps ST :

PROPOSITION 6.2. $S_*T_* = (ST)_*$.

PROOF. We check that on pure wedges

$$S_*T_*(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k) = S_*((T\mathbf{v}_1) \wedge \cdots \wedge (T\mathbf{v}_k)) = (ST\mathbf{v}_1) \wedge \cdots \wedge (ST\mathbf{v}_k) = (ST)_*(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k)$$

\square

7. The Determinant

Now, let $n = \dim V$, and recall that $\dim \bigwedge^n V = 1$. Thus, if $T : V \rightarrow V$ is a linear map, then the induced linear map

$$T_* : \bigwedge^n V \longrightarrow \bigwedge^n V$$

must be multiplication by a scalar. Namely, if $\omega \in \bigwedge^n V$ is any vector, then

$$T_*\omega = d\omega,$$

for some scalar d .

DEFINITION 7.1. Let V be an n -dimensional vector space. The **determinant** of a linear map T is the unique scalar in \mathbb{F} , called $\det T$, such that

$$T_*\omega = \det T \cdot \omega$$

for any $\omega \in \bigwedge^n V$.

Namely, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ is any basis of V , then $\det T$ is the unique scalar such that

$$(T\mathbf{v}_1) \wedge \cdots \wedge (T\mathbf{v}_n) = (\det T)(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n).$$

REMARK 7.1. In light of the analogy in §4.1, one can think of the determinant as the relative signed volume increase of an n -parallelepiped in V after applying of T .

Thanks to the way it has been defined, the determinant satisfies some key properties.

1. **Independence of basis:** The definition above is manifestly independent of choice of basis, but let us explain why. Let's suppose that

$$T_*(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n) = \det T(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n).$$

Suppose $\mathbf{w}_1, \dots, \mathbf{w}_n$ is another basis of V . Then, since $\dim \bigwedge^n V = 1$, $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_n = c\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n$ for some c . Thus,

$$\begin{aligned} T_*(\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_n) &= T_*(c\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n) \\ &= cT_*(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n) \\ &= c(\det T)(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n) \\ &= (\det T)(\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_n), \end{aligned}$$

as desired, where we have used the fact that T_* is linear.

2. **Multiplicativity.** Let $S, T : V \rightarrow V$ be two linear maps, and pick a vector $\omega \in \bigwedge^n V$. Then, note that

$$(ST)_*(\omega) = S_*(T_*\omega) = S_*(\det T \cdot \omega) = (\det S)(\det T)\omega,$$

hence

$$(11) \quad \det(ST) = \det S \cdot \det T = \det(TS).$$

Here, we used linearity and the fact that $(ST)_* = S_*T_*$.

3. Relationship to invertibility

PROPOSITION 7.1. If $\dim V = n$ and $T \in \mathcal{L}(V)$, then $\det(T) \neq 0$ if and only if T is invertible.

PROOF. $\det T \neq 0$ if and only if, for some basis $\mathbf{v}_1, \dots, \mathbf{v}_n$, $(T\mathbf{v}_1) \wedge \cdots \wedge (T\mathbf{v}_n)$ is non-zero, which occurs if and only if the list $(T\mathbf{v}_1, \dots, T\mathbf{v}_n)$ is linearly independent. This occurs if and only if T is an isomorphism, i.e. T is invertible. □

EXAMPLE 7.1 (Determinants in dimension 2). Suppose the dimension of V is 2, and $T \in \mathcal{L}(V)$. Let $\underline{\mathbf{v}} = (\mathbf{v}_1, \mathbf{v}_2)$ be a basis of V , and suppose that with respect to this basis, the matrix of T is

$$\mathcal{M}(T, \underline{\mathbf{v}}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

That is, suppose T has the following effect on the basis $\underline{\mathbf{v}}$:

$$\begin{aligned} \mathbf{v}_1 &\mapsto a\mathbf{v}_1 + c\mathbf{v}_2 \\ \mathbf{v}_2 &\mapsto b\mathbf{v}_1 + d\mathbf{v}_2 \end{aligned}$$

Then, we can calculate the induced map on the 1-dimensional vector space $\bigwedge^2 V$ in terms of the basis $(\mathbf{v}_1 \wedge \mathbf{v}_2)$:

$$\begin{aligned} T_*(\mathbf{v}_1 \wedge \mathbf{v}_2) &= T(\mathbf{v}_1) \wedge T(\mathbf{v}_2) \\ &= (a\mathbf{v}_1 + c\mathbf{v}_2) \wedge (b\mathbf{v}_1 + d\mathbf{v}_2) \\ &= ab\mathbf{v}_1 \wedge \mathbf{v}_1 + ad\mathbf{v}_1 \wedge \mathbf{v}_2 + cb\mathbf{v}_2 \wedge \mathbf{v}_1 + cd\mathbf{v}_2 \wedge \mathbf{v}_2 \\ &= ad\mathbf{v}_1 \wedge \mathbf{v}_2 + cb(-\mathbf{v}_1 \wedge \mathbf{v}_2) \\ &= (ad - bc)\mathbf{v}_1 \wedge \mathbf{v}_2, \end{aligned}$$

where the fourth equality used the fact that $\mathbf{v} \wedge \mathbf{v} = 0$ and $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$ for any \mathbf{v}, \mathbf{w} . Thus, by the definition of determinant, $\det(T) = ad - bc$. This agrees with the 2×2 determinant of the matrix $\mathcal{M}(T, \underline{\mathbf{v}})$ that you may be familiar with.

One can imagine deriving similarly explicit formulas in any dimension (this is of course easier in low dimensions!). Using identical reasoning, one can derive the formula for the determinant of an $n \times n$ matrix that is often presented in a first linear algebra class, in terms of “row-expansion.”

EXAMPLE 7.2 (The identity). Let $I : V \rightarrow V$ be the identity map. Then note that, for any basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V ,

$$I_*(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_n) = (I\mathbf{v}_1) \wedge \dots \wedge (I\mathbf{v}_n) = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_n;$$

hence

$$\det(I) = 1.$$

EXAMPLE 7.3 (Determinants of diagonalizable operators). Suppose $T : V \rightarrow V$ has an eigenbasis $\mathbf{v}_1, \dots, \mathbf{v}_n$ with associated eigenvalues $\lambda_1, \dots, \lambda_n$. Then, we can use this basis to calculate the determinant:

$$\begin{aligned} (T\mathbf{v}_1) \wedge \dots \wedge (T\mathbf{v}_n) &= (\lambda_1\mathbf{v}_1) \wedge \dots \wedge (\lambda_n\mathbf{v}_n) \\ &= (\lambda_1 \dots \lambda_n)\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_n; \end{aligned}$$

hence the determinant of T is the product of its eigenvalues with multiplicity, $\lambda_1 \dots \lambda_n$.

EXAMPLE 7.4 (Determinants of operators with upper triangular matrices). Suppose there exists a basis $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ for which the matrix of T is upper triangular. If $\lambda_1, \dots, \lambda_n$ are the diagonal entries of this matrix (thus, the eigenvalues of T with multiplicity), upper triangularity means that for each $k = 1, \dots, n$

$$T\mathbf{v}_k = \lambda_k\mathbf{v}_k + \mathbf{u}_{k-1}, \text{ for some } \mathbf{u}_{k-1} \in \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{k-1})$$

(where $\mathbf{u}_0 = 0$). Thus, with respect to this basis, we see that

$$T_*(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n) = (\lambda_1 \mathbf{v}_1) \wedge (\lambda_2 \mathbf{v}_2 + \mathbf{u}_1) \wedge \cdots \wedge (\lambda_k \mathbf{v}_k + \mathbf{u}_{k-1}).$$

Now, because $\mathbf{u}_1 \in \text{span}(\mathbf{v}_1)$, we see that $\lambda_1 \mathbf{v}_1 \wedge \mathbf{u}_1 \wedge \cdots = 0$ using the wedge independence lemma, so this expression equals

$$(12) \quad (\lambda_1 \mathbf{v}_1) \wedge (\lambda_2 \mathbf{v}_2) \wedge (\lambda_3 \mathbf{v}_3 + \mathbf{u}_2) \wedge \cdots \wedge (\lambda_k \mathbf{v}_k + \mathbf{u}_{k-1})$$

Inductively, suppose we have shown that this pure wedge is equal to

$$(13) \quad (\lambda_1 \mathbf{v}_1) \wedge \cdots \wedge (\lambda_{j-1} \mathbf{v}_{j-1}) \wedge (\lambda_j \mathbf{v}_j + \mathbf{u}_{j-1}) \wedge \cdots \wedge (\lambda_k \mathbf{v}_k + \mathbf{u}_{k-1})$$

Then, note that since $\mathbf{u}_{j-1} \in \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{j-1})$, the collection $(\lambda_1 \mathbf{v}_1, \dots, \lambda_{j-1} \mathbf{v}_{j-1}, \mathbf{u}_{j-1})$ is linearly dependent, so

$$(\lambda_1 \mathbf{v}_1) \wedge \cdots \wedge (\lambda_{j-1} \mathbf{v}_{j-1}) \wedge (\mathbf{u}_{j-1}) \wedge \cdots = 0.$$

Hence, the expression (13) is equal to

$$(14) \quad (\lambda_1 \mathbf{v}_1) \wedge \cdots \wedge (\lambda_{j-1} \mathbf{v}_{j-1}) \wedge (\lambda_j \mathbf{v}_j) \wedge (\lambda_{j+1} \mathbf{v}_{j+1} + \mathbf{u}_j) \wedge \cdots \wedge (\lambda_k \mathbf{v}_k + \mathbf{u}_{k-1}).$$

Inductively, this implies that the initial expression (12) equals

$$\lambda_1 \mathbf{v}_1 \wedge \cdots \wedge \lambda_n \mathbf{v}_n = (\lambda_1 \cdots \lambda_n) \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n,$$

so once more, the determinant of T is the product of the diagonal values of this matrix.

8. The characteristic polynomial revisited

With the determinant in hand, we can define the characteristic polynomial of a linear map $T : V \rightarrow V$ in a basis free fashion.

DEFINITION 8.1. *The **characteristic polynomial** of a map $T : V \rightarrow V$ is the polynomial*

$$q_T(z) = \det(zI - T).$$

This definition has the advantage that it can be computed using any basis. Moreover, thanks to the properties we have already developed,

PROPOSITION 8.1. *If $q_T(\lambda) = 0$, then λ is an eigenvalue of T .*

PROOF. If $q_T(\lambda) = \det(\lambda I - T) = 0$, then $\lambda I - T$ is not invertible, so $T - \lambda I$ is not invertible, i.e. λ is an eigenvalue of T . \square

Using a basis for which the matrix of T is upper triangular, one can verify that this definition of characteristic polynomial agrees with the definition we have already seen. However, this definition has the advantage of not requiring us to find a nice basis in order to calculate.

EXAMPLE 8.1 (Characteristic polynomial in dimension 2). Suppose once more that $\dim V = 2$, and $(\mathbf{v}_1, \mathbf{v}_2)$ is a basis for which $T : V \rightarrow V$ acts as:

$$\begin{aligned} \mathbf{v}_1 &\mapsto a\mathbf{v}_1 + c\mathbf{v}_2 \\ \mathbf{v}_2 &\mapsto b\mathbf{v}_1 + d\mathbf{v}_2 \end{aligned}$$

Then, we can compute the characteristic polynomial of T , $\det(zI - T)$, as follows: observe that

$$\begin{aligned}(zI - T)_*(\mathbf{v}_1 \wedge \mathbf{v}_2) &= (zI - T)(\mathbf{v}_1) \wedge (zI - T)(\mathbf{v}_2) \\ &= (z\mathbf{v}_1 - (a\mathbf{v}_1 + c\mathbf{v}_2)) \wedge (z\mathbf{v}_2 - (b\mathbf{v}_1 + d\mathbf{v}_2)) \\ &= (z - a)(-b)\mathbf{v}_1 \wedge \mathbf{v}_1 + (z - a)(z - d)\mathbf{v}_1 \wedge \mathbf{v}_2 + (cb)\mathbf{v}_2 \wedge \mathbf{v}_1 + (-c)(z - d)\mathbf{v}_2 \wedge \mathbf{v}_2 \\ &= (z - a)(z - d)\mathbf{v}_1 \wedge \mathbf{v}_2 + cb(-\mathbf{v}_1 \wedge \mathbf{v}_2) \\ &= (z^2 - (a + d)z + (ad - bc))\mathbf{v}_1 \wedge \mathbf{v}_2.\end{aligned}$$

Thus, the characteristic polynomial of T is

$$q_T(z) = z^2 - (a + d)z + (ad - bc).$$