This is a closed-book, no notes exam. Unless otherwise indicated, you should prove each of your answers. Your grade will reflect the quality of your exposition as well as the correctness of your argument. You may use results proved in the text or in class, but you must clearly state any result before applying it to your problem.

1. Let $d : \mathcal{P}(2) \rightarrow \mathcal{P}(2)$ be the linear map defined by differentiation. Find the matrix for d with respect to the indicated basis:
$$\mathcal{P}(2) = span\{1 + x, x + x^2, 1 + x^2\}.$$

2. Consider an operator T which is represented by the matrix
$$M_T = \begin{bmatrix} 2 & * & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
   (a) Assign values to the starred entries so that the degrees of the minimal and characteristic polynomials of T are the same.
   (b) Assign values to the starred entries so that the degrees of the minimal and characteristic polynomials of T are different.

3. Suppose that T is an operator on a finite-dimensional complex vector space V. Let $\mathbf{v}$ be an eigenvector with eigenvalue λ, and let $\{\mathbf{w_1}, \mathbf{w_2}, ...\mathbf{w_k}\}$ be linearly independent generalized eigenvectors corresponding to the eigenvalue μ, where $\mu \neq \lambda$. Show that $\{\mathbf{v}, \mathbf{w_1}, ...\mathbf{w_k}\}$ are linearly independent.

4. Suppose that $T \in \mathcal{L}(V, V)$ is an operator on a finite-dimensional inner product space V.
   (a) Show that T is injective if and only if T^* is bijective.
   (b) Describe the eigenvalues of T^* in terms of the eigenvalues of T.

5. Suppose that the identity operator on $\mathbb{R}^n$ can be written as $I = A + B$, where $A^2 = A$, $B^2 = B$.
   (a) Show that $AB = 0$.
   (b) Let $V = \text{range}\, A$ and $W = \text{range}\, B$. Show that $\mathbb{R}^n = V \oplus W$.

6. Suppose that V is a finite-dimensional complex vector space, and suppose further that 0 is the only eigenvalue of $T \in \mathcal{L}(V, V)$. Show that T must be nilpotent.

7. (a) State the Spectral Theorem for finite-dimensional complex inner product spaces.
   (b) Prove the theorem stated in Part 7a.

8. Prove that if S is an isometry on a real inner product space V, then
$$< S\mathbf{v}, S\mathbf{w} >=< \mathbf{v}, \mathbf{w} > \text{ for all } \mathbf{v}, \mathbf{w} \in V.$$