Math 113 Final
Spring, 2011
Part I

Please print your name here:

This is a closed-book, no notes exam. Please sign here to indicate your acceptance of the Stanford Honor Code:

Prove each of your answers. You may use results proved in the text or in class to support your argument, but you must clearly state the result before applying it to your problem. (Of course, if you are asked to prove a result presented previously, it's not sufficient to simply cite it.) Your grade will reflect the quality of your exposition as well as the correctness of your argument.

Question	Score	Maximum
1		10
2		15
3		10
Bonus		5
Total		35

1. (10) Let V be a finite-dimensional vector space, and let $T \in \mathcal{L}(V)$. Prove **one** of the statements below. (Credit will only be given for one proof; please clearly indicate which statement you are proving.)

   (a) If $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}$ are eigenvectors of T corresponding to distinct eigenvalues, then $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}$ are linearly independent.

   (b) Suppose $S_i \in \mathcal{L}(V)$ for $i = 1, 2, 3, \ldots$. Show that if each S_i is injective, then for every $n \in \mathbb{N}$, the composition $S_1 S_2 \ldots S_n$ is injective.

2. (15) Let V be a finite-dimensional complex vector space and let $T \in \mathcal{L}(V)$ have distinct eigenvalues $\lambda_1, \ldots \lambda_m$. Let $p(z) \in \mathcal{P}(\mathbb{C})$ be the polynomial

$$p(z) = (z - \lambda_1)(z - \lambda_2) \ldots (z - \lambda_m).$$

(a) Prove that if $\mathbf{v}$ is an eigenvector of T, then $p(T)\mathbf{v} = \mathbf{0}$.

(b) Prove that if $\mathbf{v}$ is a generalized eigenvector of T which is not an eigenvector, then $p(T)\mathbf{v} \neq \mathbf{0}$.

(c) Prove that $p(T) = 0$ if and only if p is the minimal polynomial of T .

3. (10) Suppose that N is a nilpotent operator on a vector space V and let $\mathbf{v} \in V$. Show that if $\{\mathbf{v}, N\mathbf{v}, N^2\mathbf{v}, N^3\mathbf{v}\}$ is linearly independent and $\{\mathbf{v}, N\mathbf{v}, N^2\mathbf{v}, N^3\mathbf{v}, N^4\mathbf{v}\}$ is linearly dependent, then $N^4\mathbf{v} = \mathbf{0}$.

4. (5) **Bonus Question** (This question is not worth many points. Please work on the rest of the exam and only come back to this if you've done everything else.)

   Suppose that A and B are commuting matrices representing normal operators on a finite dimensional complex inner product space V. Show that there exists a matrix U such that the matrices UAU^* and UBU^* are both diagonal.

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Part II

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Question	Score	Maximum
5		10
6		10
7		10
8		10
Total		40

5. (10) Give an example of two matrices with the same minimal and characteristic polynomials which are not conjugate to one another.

   (Hint: Two matrices are conjugate if and only if they have the same Jordan form, up to reordering the blocks.)

6. (10) Let V be a finite-dimensional inner product space. Prove **one** of the statements below. (Credit will only be given for **one** proof; please clearly indicate which statement you are proving.)

   (a) If $P^2 = P$ and P is self-adjoint, then P is an orthogonal projection.

   (b) If $P^2 = P$ and $||P\mathbf{v}|| \leq ||\mathbf{v}||$ for all $\mathbf{v} \in V$, then P is an orthogonal projection.

7. (10) An operator $T \in \mathcal{L}(V)$ is *skew-adjoint* if $T^* = -T$. Describe the eigenvalues of a skew-adjoint operator as completely as you can. Prove your answer.

8. (10)

(a) A matrix is *symmetric* if $M^T = M$. Give an example of symmetric matrix with real entries such that i is an eigenvalue, or explain why no such matrix exists.

(b) Let $\mathcal{B} = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$, and suppose that with respect to $\mathcal{B}$, $T \in \mathcal{L}(\mathbb{R}^2)$ is represented by

$$\mathcal{M}_{\mathcal{B}}(T) = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

Find a matrix representing T^*. (Specify the basis you're using to represent T^*).

Note: the version offered on June 8 had an additional part to this problem. As written, the statement was only true in the case that the given operator was a projection.