Math 113 midterm, 2/7/11, 7pm-9pm.

Open books, open notes. No calculators. No computers, cell phones, or other internet capable devices.

1. (10p) Let V and W be finite dimensional vector spaces. Prove that there exists an injective linear map $T : V \to W$ if and only if $\dim(V) \leq \dim(W)$.

2. (10p) Let $T \in \mathcal{L}(V)$.

(i) Prove that if $T^2 = I$, then $V = \text{Null}(T + I) \oplus \text{Null}(T - I)$.

(ii) Prove that if $V = \text{Null}(T + I) + \text{Null}(T - I)$, then $T^2 = I$.

3. (10p) Let $T \in \mathcal{L}(\mathbb{C}^2)$ be the map given by $T(x, y) = (2x - y, x)$.

(i) Prove that 1 is an eigenvalue of T and find a basis for $\text{Null}(T - I)$.

(ii) Does T have other eigenvalues? (as usual, you must prove your answer).

4. (10p) Let $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$.

(i) Prove that if λ is an eigenvalue of T, then $p(\lambda)$ is an eigenvalue of $p(T)$.

(ii) Now assume $p(T) = 0$. Prove that all eigenvalues of T are roots of p.

5. (10p) Let $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ be the linear map given by $(Tp)(t) = p(t + 1)$.

(i) Find the matrix of T with respect to the basis $(1, t, t^2)$ of $\mathcal{P}_2(\mathbb{R})$.

(ii) Let A denote the matrix of T from *(i)*. Prove that there exists a matrix $B \in \text{Mat}(3, 3, \mathbb{R})$ such that $AB = BA = \mathcal{M}(I)$.

6. (20p) Let V be finite dimensional and let $T \in \mathcal{L}(V)$. For $k \geq 0$, set $U_k = \text{Null}(T^k)$.

(i) Prove that $U_k \subseteq U_{k+1}$ for all $k \geq 0$.

(ii) Prove that U_k is invariant under T for all $k \geq 0$.

(iii) Let $n = \dim(V)$. Prove that if $U_n \neq V$, then $U_{k-1} = U_k$ for some $k \leq n$.

(iv) Prove that if $U_{k-1} = U_k$, then $U_k = U_{k+1}$.

(v) Prove that if $U_n \neq V$, then $U_k \neq V$ for all $k \geq n$.

(vi) Deduce that if $T^k = 0$ for some k, then $T^n = 0$ for $n = \dim(V)$.