MIDTERM SOLUTION

Notation: We use $\mathbb{N} = \{0, 1, 2, \ldots\}$ to denote the set of all positive integers.

1. Solution: It follows from corollary 3.5 that if there is an injective linear map $V \to W$, then $\dim(V) \leq \dim(W)$.

We then prove suthe converse. Let $(e_1, \ldots, e_m)$ be a basis for V and $(f_1, \ldots, f_n)$ be a basis for W, with $m \leq n$. We define a linear map $T : V \to W$ by $T(c_1 e_1 + \cdots + c_m e_m) = c_1 f_1 + \cdots + c_m f_m$. We prove that T is injective. Suppose $T(c_1 e_1 + \cdots + c_m e_m) = 0$, then $c_1 f_1 + \cdots + c_m f_m = 0$. Since $(f_1, \ldots, f_m)$ is linearly independent, it follows that $c_1 = \cdots = c_m = 0$. Therefore $\mathrm{null}(T) = \{0\}$, T is injective.

To sum up, the proof is complete.

2. Solution: We first show that if $T^2 = I$, then $V = \mathrm{null}\ (T + I) \oplus \mathrm{null}\ (T - I)$. For any vector $v \in V$, consider the decomposition $v = \frac{1}{2}(v - Tv) + \frac{1}{2}(v + Tv)$. Using the condition that $T^2 = I$, for the first component, we have $(T + I)(\frac{1}{2}(v - Tv)) = \frac{1}{2}(I - T^2)v = 0$, i.e. $\frac{1}{2}(v - Tv) \in \mathrm{null}\ (T + I)$. For the second component, we have $(T - I)(\frac{1}{2}(v + Tv)) = \frac{1}{2}(T^2 - I)v = 0$, i.e. $\frac{1}{2}(v + Tv) \in \mathrm{null}\ (T - I)$. Therefore it follows that $V = \mathrm{null}\ (T + I) + \mathrm{null}\ (T - I)$. To show that the sum is direct, we only need to show that $\mathrm{null}\ (T+I) \cap \mathrm{null}\ (T-I) = \{0\}$. Suppose $v \in \mathrm{null}\ (T+I) \cap \mathrm{null}\ (T-I)$. Then $(T+I)v = 0$ and $(T-I)v = 0$, i.e. $Tv + v = 0, Tv - v = 0$. Hence $v = \frac{1}{2}((Tv + v) - (Tv - v)) = 0$. This shows that $\mathrm{null}\ (T + I) \cap \mathrm{null}\ (T - I) = \{0\}$, as desired. To sum up, we have $V = \mathrm{null}\ (T + I) \oplus \mathrm{null}\ (T - I)$.

We then show that if $V = \mathrm{null}\ (T + I) + \mathrm{null}\ (T - I)$, then $T^2 = I$. Take $v \in V$, since $V = \mathrm{null}\ (T + I) + \mathrm{null}\ (T - I)$, there exists $v_1 \in \mathrm{null}\ (T + I)$ and $v_2 \in \mathrm{null}\ (T - I)$, such that $v = v_1 + v_2$. Since $v_1 \in \mathrm{null}\ (T + I)$, we have $(T^2 - I)v_1 = (T - I)(T + I)v_1 = (T - I)0 = 0$, i.e. $T^2 v_1 = v_1$. Similarly, $v_2 \in \mathrm{null}\ (T - I)$ implies that $(T^2 - I)v_2 = (T + I)(T - I)v_2 = (T + I)0 = 0$, i.e. $T^2 v_2 = v_2$. It follows that $T^2 v = T^2 v_1 + T^2 v_2 = v_1 + v_2 = v$, for any vector $v \in V$. Hence $T^2 = I$, as desired.

To sum up, the proof is complete.

3. Solution: (i) It follows from the definition of T that $(T - I)(x, y) = (x - y, x - y)$. Therefore $(x, y) \in \mathrm{null}(T - I)$ if and only if $x - y = 0, x - y = 0$, i.e. $x = y$. Therefore $((1, 1))$ is a basis for $\mathrm{null}(T - I)$ and 1 is an eigenvalue of T.

(ii) The answer is negative. We prove that for any $\lambda \neq 1$, λ is not an eigenvalue for T. Proof by contradiction, suppose $\lambda \neq 1$ is an eigenvalue for T, then there exists a vector $(x, y) \neq (0, 0)$ such that $T(x, y) = \lambda(x, y)$, i.e. $(2x - y, x) = (\lambda x, \lambda y)$. Rearranging, we get $(2 - \lambda)x - y = 0, x = \lambda y$. Substitute $x = \lambda y$ into the first equation to get $(2\lambda - \lambda^2 - 1)y = 0$, i.e. $-(1 - \lambda)^2 y = 0$. Since $\lambda \neq 1$, we must have $y = 0$. It then follows from $x = \lambda y$ that $x = 0$. Hence $(x, y) = (0, 0)$, a contradiction. This completes the proof.

4. Solution: (i) Let $p(z) = a_n z^n + \cdots + a_0$ be a polynomial in $\mathbb{F}$. Let λ be an eigenvalue of T, $v \neq 0$ be a corresponding eigenvector, i.e. $Tv = \lambda v$. An easy induction argument shows that $T^k v = \lambda^k v$ for any $k \in \mathbb{N}$. Then $p(T)v = \sum_{i=0}^{n} a_i T^i v = \sum_{i=0}^{n} a_i \lambda^i v = p(\lambda)v$. Therefore $p(\lambda)$ is an eigenvalue with a corresponding eigenvector $v \neq 0$. This completes the proof.

(ii) Suppose $p(z) = a_n z^n + \cdots + a_0$ is a polynomial in $\mathbb{F}$ such that $p(T) = 0$ and let λ be an eigenvalue of T with a corresponding eigenvector $v \neq 0$. Using the fact that $\lambda^k v = T^k v$ for all $k \in \mathbb{N}$, we get $p(\lambda)v = \sum_{i=0}^{n} a_i \lambda^i v = \sum_{i=0}^{n} a_i T^i v = p(T)v = 0v = 0$. Since $v \neq 0$, we must have $p(\lambda) = 0$. Hence λ is a root of p, which completes the proof.

5. Solution: (i) It follows from the definition of T that $T(t^2) = (t+1)^2 = t^2 + 2t + 1, T(t) = t + 1, T(1) = 1$. Therefore the matrix for T with respect to the basis $(1, t, t^2)$ is given by $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(ii) Let $S \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ be given by $(Sp)(t) = p(t-1)$. Then clearly $ST = TS = I$, so $B = \mathcal{M}(S)$ satisfies $AB = BA = \mathcal{M}(I)$, by formula 3.11.

6. Solution: (i) It suffices to show that for any $v \in U_k$, we have $v \in U_{k+1}$. The condition that $v \in U_k$ means that $T^k v = 0$. Therefore $T^{k+1} v = T(T^k v) = T0 = 0$, hence $v \in U_{k+1}$. This completes the proof.

(ii) It suffices to show that for any $v \in U_k$, we have $Tv \in U_k$. The condition that $v \in U_k$ means that $T^k v = 0$. Therefore $T^k(Tv) = T^{k+1} v = T(T^k v) = T0 = 0$, hence $Tv \in U_k$. This completes the proof.

(iii) It follows from Proposition 2.15, Exercise 2-11 and the condition $U_n \neq V$ that $\dim U_n \leq n - 1$. Using part (i), it follows that $U_0 \subset U_1 \subset \cdots \subset U_n$. Therefore it follows from the above observation and Homework 2-8 that $U_{k-1} = U_k$ for some $1 \leq k \leq n$. This completes the proof.

(iv) By part (i), we have $U_k \subset U_{k+1}$, so it suffices to show that $U_{k+1} \subset U_k$. Let v be a vector in U_{k+1}, then $T^{k+1} v = 0$, i.e. $T^k(Tv) = 0$. Hence $Tv \in U_k$. By the condition that $U_k = U_{k-1}$ we have $Tv \in U_{k-1}$. It follows from the definition of U_{k-1} that $T^{k-1}(Tv) = 0$, i.e. $T^k v = 0$, $v \in U_k$ for any vector v in U_{k+1}. This completes the proof.

(v) It follows from part (iii) that $U_k = U_{k-1}$ for some $k \leq n$. Therefore by part (iv), $U_r = U_{k-1}$ for any $r \geq k - 1$. Since $k \leq n$, we have $U_k = U_n$ for any $k \geq n$. Using the condition that $U_n \neq V$, we conclude that $U_k \neq V$ for any $k \geq n$.

(vi) Proof by contradiction. Suppose $T^n \neq 0$, then $U_n \neq V$. It follows from part (v) that $U_k \neq V$ for any $k \geq n$. Using part (i), we conclude that $U_k \neq V$ for any $k \geq 0$. It follows from the definition of U_k that $T^k \neq 0$ for any k, a contradiction. This completes the proof.

References

[A] S. Axler, *Linear Algebra Done Right.*