Math 171 Homework 1

(due April 8)

Problem 5.4.
Show that if X is a nonempty subset of $\mathbb{R}$ which is bounded above then

$$\sup X = -\inf(-X).$$

Solution:
We need to show that: (a) $-\inf(-X)$ is an upper bound of X and (b) that if b is an upper bound of X then $-\inf(-X) \leq b$.

(a) For every $x \in X$ we have that $-x \in -X$, so

$$\inf(-X) \leq -x.$$

Multiplying the equality above by -1 (see Theorem 4.2v in Johnsonbaugh and Pfaffenberger):

$$-\inf(-X) \geq x.$$

Since the above inequality holds for every $x \in X$, $-\inf(-X)$ is an upper bound of X.

(b) Assume b is an upper bound of X. For every $y \in -X$ we have that $-y \in X$, so

$$-y \leq b.$$

Multiplying the inequality above by -1 we get that

$$-b \leq y.$$

Since this holds for every $y \in -X$ we have that $-b$ is a lower bound of $-X$. Thus, $-X$ is bounded below and

$$-b \leq \inf(-X).$$

Multiplying the above inequality by -1 gives us the desired result:

$$-\inf(-X) \leq b.$$

Problem 5.7.
Let X and Y be subsets of $\mathbb{R}$ with least upper bounds a and b, respectively. Prove that $a+b$ is the least upper bound of the set

$$X + Y := \{x + y \mid x \in X, y \in Y\}.$$

Solution:

(a) We show that $a+b$ is an upper bound of $X+Y$. Given any element z of $X+Y$, by definition of $X+Y$, z can be written as

$$z = x + y$$

for some $x \in X$ and $y \in Y$. Since a and b are upper bounds of X and Y, respectively, we have that

$$x \leq a \qquad \text{and} \qquad y \leq b.$$

Hence $x+y \leq a+b$, which is the same as $z \leq a+b$. Since z was an arbitrary element of $X+Y$, we have that $a+b$ is an upper bound of $X+Y$.

(b) Next we show that any upper bound of $X+Y$ is greater or equal to $a+b$ by contradiction. Assume that c is an upperbound of $X+Y$ satisfying

$$c < a + b.$$

Let $\epsilon := a+b-c > 0$. Since a is the *least* upper bound of X, $a - \epsilon/2$ (which is smaller than a) is *not* an upper bound of X, so there exists $x \in X$ such that

$$a - \frac{\epsilon}{2} < x.$$

Similarly, there exists $y \in Y$ such that

$$b - \frac{\epsilon}{2} < y.$$

Hence,

$$\left(a - \frac{\epsilon}{2}\right) + \left(b - \frac{\epsilon}{2}\right) < x + y.$$

After simplifying we see that the left hand side of the inequality above is equal to c. Thus,

$$c < x + y$$

contradicting the assumption that c is an upper bound of $X+Y$.

Thus, $a+b$ is the least upper bound of $X+Y$.

Problem 9.10.
Prove that the plane is *not* the union of a countable family of straight lines.
Solution:
Assume the contrary – that the plane is the union of a countable family $\{L_n\}_{n\in\mathbb{N}}$ of straight lines.
Consider a different family of lines - the family of horizontal lines $\{P_r\}_{r\in\mathbb{R}}$ – for each real number r we get a different horizontal line $P_r = \{y = r\}$.
The family $\{P_r\}_{r\in\mathbb{R}}$ is uncountable since it is equivalent to $\mathbb{R}$. Hence, there exists a line P_r in this family which is not one of the lines in $\{L_n\}$.
By assumption every point (x, r) in P_r is covered by at least one of the lines L_n. For each x choose such an $n(x)$ such that $(x, r) \in L_{n(x)}$. Since two distinct lines intersect in at most

one point all of these $n(x)'s$ are distinct: if P_r intersect L_n in (x, r) it can't also intersect L_n at (x', r) with $x' \neq x$.
On one hand the set $\{L_{n(x)} \mid x \in \mathbb{R}\}$ is uncountable, being equivalent to $\mathbb{R}$ (for each x in $\mathbb{R}$ we get a distinct point (x, r) of P_r and in turn a distinct line $L_{n(x)}$ passing through that point).
On the other hand the set $\{L_{n(x)} \mid x \in \mathbb{R}\}$ is countable being a subset of a countable set $\{L_n\}_{n \in \mathbb{N}}$.
We get a contradiction. Hence, our initial assumption that the plane can be covered by countably many lines is wrong.

Problem 10.5.
Prove that

$$\lim_{n \to \infty} \frac{n}{n+2} = 1.$$

using the definition of a limit (Definition 10.2).
Solution:
We have that

$$\frac{n}{n+2} - 1 = \frac{n - (n+2)}{n+2} = -\frac{2}{n+2}.$$

Hence,

$$\left| \frac{n}{n+2} - 1 \right| = \frac{2}{n+2}$$

for $n \geq 0$.
Fix an arbitrary $\epsilon > 0$. We are be looking for an N such that for every $n \geq N$ we will have $|\frac{n}{n+2} - 1| < \epsilon$. Thus, we are trying to solve

$$\frac{2}{n+2} < \epsilon.$$

The latter inequality is equivalent to

$$\frac{2}{\epsilon} < n + 2$$

(via multiplying both sides by $(n+2)$ and dividing by ϵ). Thus, if we choose N to be any integer greater than $2/\epsilon$ we will have that

$$\left| \frac{n}{n+2} - 1 \right| < \epsilon,$$

for every $n \geq N$ as desired.

Problem 13.2.
Let $\{a_n\}$ be a sequence with limit 0. Prove that

$$\lim_{n \to \infty} (-1)^n a_n = 0.$$

Solution:

Fix an arbitrary $\epsilon > 0$. By assumption, there exists an N such that for every $n \geq N$ we have that

$$|a_n - 0| < \epsilon,$$

which is the same as $|a_n| < \epsilon$.
Therefore, for every $n \geq N$ we have that

$$|(-1)^n a_n - 0| = |(-1)^n a_n| = |a_n| < \epsilon,$$

as desired.

Problem 16.7.
Prove that every convergent sequence $\{a_n\}$ has a monotone subsequence.
Solution:
Let say l is the limit of $\{a_n\}$. Because there are infinitely terms in a sequence then at least one of the following statements holds true:

- there are infinitely many n for which $a_n = l$;
- there are infinitely many n for which $a_n > l$;
- there are infinitely many n for which $a_n < l$.

We show that each of the statements above implies existence of a monotone subsequence.

- Assume there are in are infinitely many n for which $a_n = l$. Then we can discard all n for which $a_n \neq l$ and get a subsequence of $\{a_n\}$ which is constant, and in particular, both decreasing and increasing. Note, that the assumption was necessary in order to get an infinite subsequence rather than a finite set of terms.
- Assume there are infinitely many n for which $a_n > l$. Similarly to the previous part, start by throwing away all n for which $a_n \leq l$. Then we are left with a subsequence of $\{a_n\}$ converging to l all of whose terms are greater than n. Let's call this subsequence $\{b_n\}$ in order to avoid nested subscripts.

  We then recursively choose a decreasing subsequence $\{b_{n_k}\}$ of $\{b_n\}$ (which is going to serve a the desired monotone subsequence of $\{a_n\}$).

  Assume we already chose $\{n_k\}$ for all $k \leq m$ such that $b_{n_k} < b_{n_{k-1}}$. Then all we need to find for the recursive step is n_{m+1} such that

  - $n_{m+1} > n_m$
  - $b_{n_{m+1}} < b_{n_m}$

  Since $b_{n_m} > l$ and $\{b_n\}$ converges to l we can use $b_{n_m} - l$ as ϵ in the definition of the limit to get that there exists N such that for all $n \geq N$ we have that

  $$|b_n - l| < b_{n_m} - l$$

  Since by construction of $\{b_n\}$, $b_n > l$ the above inequality becomes

  $$b_n - l < b_{n_m} - 1$$

  i.e. $b_n < b_{n_m}$ for all $n \geq N$. Thus, we may take n_{m+1} to be N.

- Assume there are infinitely many n for which $a_n < l$. In this case we construct an increasing subsequence following all of the steps of the previous case.

Problem 16.10.

(a) Let x and y be positive numbers. Let $a_0 := y$, and let

$$a_n := \frac{(x/a_{n-1}) + a_{n-1}}{2} \qquad \text{for } n = 1, 2, \ldots$$

Prove that $\{a_n\}$ is a decreasing sequence with limit $\sqrt{x}$.

(b) Generalize (a) to k^{th} roots.

Solution:

(a) We start by proving $a_n > 0$ by induction on n:

- the base case $n = 0$: $a_0 = y > 0$ by assumption;
- induction step, if $a_{n-1} > 0$ then $x/a_{n-1} > 0$ so $a_n > 0$.

Next we show that $\{a_n\}$ is decreasing for $n \geq 1$. We have the following chain of equivalences of inequalities

$$\begin{aligned} a_n \leq a_{n-1} &\Leftrightarrow \frac{(x/a_{n-1}) + a_{n-1}}{2} \leq a_{n-1} \\ &\Leftrightarrow \frac{x}{a_{n-1}} \leq a_{n-1} \\ &\Leftrightarrow x \leq a_{n-1}^2 \\ &\Leftrightarrow \sqrt{x} \leq a_{n-1}. \end{aligned}$$

Thus, if we prove that $a_n \geq \sqrt{x}$ for $n \geq 1$ we get that $\{a_n\}$ is decreasing for $n \geq 1$.

Note that a_n is the arithmetic mean of two terms (x/a_{n-1} and a_{n-1}) whose geometric means is $\sqrt{x}$:

$$\sqrt{x/a_{n-1} \cdot a_{n-1}} = \sqrt{x}.$$

Thus, by the AM-GM inequality $a_n \geq \sqrt{x}$.

Hence, $\{a_n\}$ is decreasing for $n \geq 1$.

Since, $\{a_n\}$ is decreasing and bounded below (by $\sqrt{x}$ as we have shown above), it converges (Theorem 16.2). Let's call its limit l.

By Lemma 14.2 applied to the constant sequence $b_n := \sqrt{x}$ and $\{a_n\}$, we have that

$$\sqrt{x} \leq l.$$

We have that

$$\lim_{n\to\infty} a_{n-1} = l$$

(because $\{a_{n-1}\}$ is that same sequence as $\{a_n\}$ up to index shift) and

$$\lim_{n\to\infty} \frac{x}{a_{n-1}} = \frac{x}{l}$$

by Lemma 12.8 and Theorem 12.3. Therefore, on one hand

$$\lim_{n\to\infty} \frac{(x/a_{n-1}) + a_{n-1}}{2} = \frac{x/l + l}{2}$$

and on the other hand

$$\lim_{n\to\infty} \frac{(x/a_{n-1}) + a_{n-1}}{2} = \lim_{n\to\infty} a_n = l.$$

We can now solve for l:

$$\frac{x/l + l}{2} = l \quad \Leftrightarrow \quad x/l = l \quad \Leftrightarrow \quad x = l^2 \quad \Leftrightarrow \quad l = \sqrt{x}$$

Thus, $\lim_{n\to\infty} a_n = \sqrt{x}$ as desired.

(b) The intuition look for a recursive definition of $\{a_n\}$ where a_n is an arithmetic mean of terms whose geometric mean equals $\sqrt[k]{x}$. It turns out the following statement is true:

Let x and y be positive numbers. Let $a_0 := y$, and let

$$a_k := \frac{(x/a_{n-1}^{k-1}) + (k-1)a_{n-1}}{k}$$

Then $\{a_n\}$ is decreasing for $n \geq 1$ and converges to $\sqrt[k]{x}$.
We prove the more general statement following the same steps as in part (a):

- We prove that a_n is positive by induction on n,
- By AM-GM inequality,
$$a_n \geq \sqrt[k]{x}$$
- We show that $a_n \leq a_{n-1}$ is equivalent to $a_{n-1} \geq \sqrt[k]{x}$ and conclude that $\{a_n\}$ is decreasing,
- Because $\{a_n\}$ is bounded and decreasing it converges to some l which satisfies
$$l = \frac{x/l + (k-1)l}{k}.$$
- Solving the latter equation for l we get that $l = \sqrt[k]{x}$.

Problem 18.5. Let $\{a_n\}$ be a sequence such that for some $\epsilon > 0$, $|a_n - a_m| \geq \epsilon$ for all $n \neq m$. Prove that $\{a_n\}$ has no convergent subsequence.

Solution:

Let $\{a_n\}$ be as above and suppose that there is a convergent subsequence $\{a_{n_k}\}$ converging to L. This means that, for $\epsilon' := \epsilon/2$, there is an $N > 0$ so that for all $k \geq N$, $|a_{n_k} - L| < \epsilon' := \epsilon/2$. Pick two natural numbers $k \neq k'$ with $k, k' \geq N$. It follows that $|a_{n_k} - L| < \epsilon'$ and $|a_{n_{k'}} - L| < \epsilon'$. Thus, by the triangle inequality, $|a_{n_k} - a_{n_{k'}}| = |a_{n_k} - L + L - a_{n_{k'}}| \leq |a_{n_k} - L| + |a_{n_{k'}} - L| < 2\epsilon' = \epsilon$. But this is a contradiction to the hypotheses on $\{a_n\}$. We conclude that $\{a_n\}$ does not have a convergent subsequence.

Problem 20.13.

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\{a_n\}$ is convergent and $\{b_n\}$ is bounded. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

and

$$\liminf_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

Solution:

Let $c_n := a_n + b_n$ and let l be the limit of a. Let $\mathcal{L}_b$ and $\mathcal{L}_c$ are the limit sets (i.e. sets of limits of all converging subsequences) of $\{b_n\}$ and $\{c_n\}$, respectively. We will show that

$$\mathcal{L}_c = \mathcal{L}_b + \{l\},$$

i.e.

$$\mathcal{L}_c = \{x + l \mid x \in L_b\}.$$

Then it will follow by Problem 5.7 that

$$\sup \mathcal{L}_c = \sup \mathcal{L}_b + l$$

which (after taking into account that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = l$ by Theorem 20.4) is equivalent to:

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Showing that $\mathcal{L}_c = \mathcal{L}_b + \{l\}$ is equivalent to showing that any real number x is in $\mathcal{L}_b$ if and only if $x + l$ is in $\mathcal{L}_c$.

- Assume $x \in \mathcal{L}_b$. By definition of $\mathcal{L}_b$ there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ that converges to x. By Theorem 11.2 the corresponding subsequence $\{a_{n_k}\}$ of $\{a_n\}$ also converges to l. Hence, by Theorem 12.2 the corresponding subsequence $c_{n_k} = a_{n_k} + b_{n_k}$ converges to $x + l$, so $x + l \in \mathcal{L}_c$.

- Conversely, assume $x + l \in \mathcal{L}_c$. Then there exists a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ that converges to $x + l$. Then by the same argument as in the previous part, $b_{n_k} = c_{n_k} - a_{n_k}$ converges to $x = (x + l) - l$, so $x \in \mathcal{L}_b$.

The statement about lim inf follows from $\mathcal{L}_c = \mathcal{L}_b + \{l\}$ and the results of Problem 5.4 and Problem 5.7.

Problem 1.

Let $\{a_n\}$ be a sequence of real numbers. We say a real number l is a *cluster point* of $\{a_n\}$ if for any $\epsilon > 0$, there are infinitely many a_n within ϵ of l. Formally, for any $\epsilon > 0$, given any $N \in \mathbb{N}$, there exists an $n \geq N$ with $|a_n - l| < \epsilon$.

(a) Show that if l is a cluster point of $\{a_n\}$, then there exists a subsequence of $\{a_n\}$ converging to l.

(b) Suppose $\mathfrak{a} := \{a_n\}$ is a bounded sequence and let $C_\mathfrak{a}$ denote the set of all cluster points of $\mathfrak{a}$. Show that $C_\mathfrak{a}$ is bounded (so its sup and inf exist), and moreover that

$$\limsup_n \{a_n\} = \sup C_\mathfrak{a}$$

$$\liminf_n \{a_n\} = \inf C_\mathfrak{a}$$

Solution:

(a) Assume l is a cluster point of $\{a_n\}$. We will recursively construct a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to l. More precisely, we recursively construct a strictly increasing sequence n_k of positive integers such that

$$|a_{n_k} - l| < \frac{1}{k}. \tag{1}$$

Assume we have constructed the first m terms of $\{n_k\}$. Then we can choose the $(m+1)^{st}$ term n_{m+1} satisfying

$$n_{m+1} >= n_m + 1 \qquad \text{and} \qquad |a_{n_{m+1}} - l| < \frac{1}{m+1}$$

using the definition of a cluster point.

Thus, we have constructed a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying (1). We show that

$$\lim_{k\to\infty} a_{n_k} = l.$$

Given $\epsilon > 0$, choose an integer $N > 1/\epsilon$. Then for every $k \geq N$ we have that

$$|a_{n_k} - l| < \frac{1}{k} < \frac{1}{N} < \epsilon.$$

Hence, $\lim_{k\to\infty} a_{n_k} = l$, as desired.

(b) Let $\mathcal{L}_{\mathfrak{a}}$ be the set of all limits of converging subsequences of $\{a_n\}$. Then we have that

$$\limsup_n \{a_n\} = \sup \mathcal{L}_{\mathfrak{a}} \qquad \text{and} \qquad \liminf_n \{a_n\} = \inf \mathcal{L}_{\mathfrak{a}}$$

(see Definition 20.1). Thus, if we are able to prove that

$$\mathcal{L}_{\mathfrak{a}} = C_{\mathfrak{a}}$$

then we are done.

By part (a) we have that $C_{\mathfrak{a}} \subset \mathcal{L}_{\mathfrak{a}}$, so we are left to prove that $\mathcal{L}_{\mathfrak{a}} \subset C_{\mathfrak{a}}$.

Let l be an arbitrary element of $\mathcal{L}_{\mathfrak{a}}$. We will prove that l is an element of $C_{\mathfrak{a}}$. By definition of $\mathcal{L}_{\mathfrak{a}}$ there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converging to L. Thus, for any $\epsilon > 0$ we can choose M such that for every $k \geq M$ we have that

$$|a_{n_k} - l| < \epsilon. \tag{2}$$

Given any N we can produce a k with $k \geq M$ and $n_k \geq N$ (the latter due to $\lim_{k\to\infty} n_k = \infty$). Thus, this k will also satisfy (2). Thus, given an arbitrary ϵ and N we have produced n (namely $n := n_k$) such that $|a_n - l| < \epsilon$ and $n \geq N$.

Problem 2.

It may be used without proof that the complex numbers $\mathbb{C}$ satisfy the axioms of a field (axioms 1-11) in Johnsonbaugh and Pfaffenberger, or F1-F6 in Prof. Simon's notes. Prove that they *don't* satisfy the axioms of an ordered field (axiom 12 in JP or O1-O2 in Prof. Simon's notes).

Solution:

We'll proceed by contradiction. Assume that $\mathbb{C}$ did satisfy the axioms of an ordered field, i.e. that there exists a subset P of $\mathbb{C}$ such that

(a) if $x, y \in P$ then $x + y, xy \in P$ and

(b) for every $x \in \mathbb{C}$ either $x = 0$, $x \in P$ or $x \in -P$ and no two of the conditions hold simultaneously.

By condition (b) exactly one of i and $-i$ is in P. Let's call this number α. Then $\alpha \in P$ and $\alpha^2 = -1$.

By condition (a) with $x = y = \alpha$ we have that $\alpha^2 = -1 \in P$. Then by condition (a) with $x = \alpha$ and $y = -1$ we have that $-\alpha \in P$. Thus, both α and $-\alpha$ are in P, thus contradicting condition (b).

Hence, $\mathbb{C}$ can't satisfy the ordered field axioms.