Math 171 Homework 2

Due Friday April 15, 2016 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Alex Zamorzaev, in his office, 380-380M (either hand your solutions directly to him or leave the solutions under his door).

Book problems: Solve Johnsonbaugh and Pfaffenberger, problems 22.7, 23.5, 24.1 (this fact was stated in class without proof), 24.2, 24.9, 25.4, 26.7, 27.2, 28.1, 19.2. Also solve:

1. More general sums: Let $E \subset \mathbb{R}$ be any set of *positive* real numbers. Let $\mathcal{F} \subset \mathcal{P}(E)$ be the set of finite subsets of E (recall that $\mathcal{P}(E)$, the *power set* of E, is the set of all subsets), and define

$$\sum_{x \in E} x := \sup_{F \in \mathcal{F}} s_F = \sup\{s_F | F \in \mathcal{F}\}. \tag{1}$$

where $s_F = \sum_{f \in F} f$ is the usual sum of the elements of the finite subset $F \subset E$.

(a) Show that $\sum_{x \in E} x < \infty$ only if E is countable.

(b) Show that if E is countably infinite and $\{x_n\}$ is an *enumeration* of E (namely, $x_i = f(i)$ for $f : \mathbb{N} \xrightarrow{\cong} E$ a bijection), then

$$\sum_{x \in E} x = \sum_{i=1}^{\infty} x_i. \tag{2}$$

2. Decimal (and base p) expansions: Let $p \in \mathbb{N} \backslash \{1\} = \{2, 3, 4, \ldots\}$ and let x be a real number with $0 < x < 1$.

(a) Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}. \tag{3}$$

(b) Moreoever, show that such a sequence $\{a_n\}$ is unique except when $x = \frac{q}{p^n}$ for another integer q; in this case, show that there are exactly two such sequences.

(c) Conversely, show that if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series (3) converges to a real number x with $0 \leq x \leq 1$.

(If $p = 10$, this $\{a_n\}$ is called the *decimal expansion* of x and gives a representation of x more familiar with from earlier math classes: "$x = 0.a_1a_2a_3a_4\ldots$"). If $p = 2$, this is called the *binary expansion*, also mentioned in class).

(d) Finally, consider the case $p = 2$. Let $S_{0,1}$ denote the set of *binary sequences*, by definition the set of all sequences $\{a_i\}_{i \in \mathbb{N}}$ where each $a_i \in \{0, 1\}$ (recall we discussed this set in class). Show using the previous two parts that there is a bijection $S_{0,1} \backslash C \cong (0, 1)$, where $C \subset S_{0,1}$ is some countable subset. Conclude that the uncountability of $S_{0,1}$ (proven in class) implies the uncountability of $\mathbb{R}$, $(0, 1)$ or any non-empty interval (a, b).