Math 171 Homework 2

(due April 15)

Problem 22.7.
Let $\{a_n\}$ and $\{b_n\}$ be sequences such that there exists an N such that for every $n \geq N$ we have that $a_n = b_n$. Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} b_n$ is convergent. If the sum $\sum_{n=1}^{\infty} a_n$ is L, find the sum of the series $\sum_{n=1}^{\infty} b_n$. (This exercise show that altering a finite number of terms of an infinite series does not affect the convergence of the series, but that it may affect the sum of the series.)
Solution:
Let $A_n := \sum_{i=1}^{n} a_n$ and $B_n := \sum_{i=1}^{n} b_n$ be the respective partial sums. We will show that B_n converges to $L + (B_N - A_N)$ (where N is the fixed integer from the statement of the problem) by proving the following lemma

Lemma 1. *For $n \geq N$ the following identity holds*

$$B_n = A_n + (B_N - A_N)$$

Proof. We prove the lemma by induction on n.
Induction basis: for $n = N$ we have that

$$B_N = A_N + (B_N - A_N).$$

Induction step: assume $B_n = A_n + (B_N - A_N)$ for an $n \geq N$. Then

$$B_{n+1} = B_n + b_n = A_n + b_{n+1} + (B_N - A_N).$$

Since $n + 1 > n \geq N$ we have that $b_{n+1} = a_{n+1}$, so

$$B_{n+1} = A_n + a_{n+1} + (B_N - A_N) = A_{n+1} + (B_N - A_N)$$

as desired. □

Assuming the lemma, Theorem 12.2 implies that

$$\lim_{n\to\infty} B_n = L + (B_N - A_N)$$

Problem 23.5. Give an example of divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.
Solution:
Let $a_n := 1$ and $b_n := -1$ for all n. Then the partial sums of $\sum_{n=1}^{\infty} a_n$ approach ∞, the partial sums of $\sum_{n=1}^{\infty} b_n$ approach $-\infty$ and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} 0$ converges to 0.
Problem 24.1. Use induction to prove that if

$$s_n = \sum_{i=1}^{n} \frac{1}{i},$$

then

$$s_{2^n} \geq \frac{n+2}{2}. \tag{1}$$

Solution:
Induction basis: for $n = 0$,

$$s_{2^0} = s_1 = 1 = \frac{0+2}{2},$$

so the equality in (1) holds.
Induction step: assume $s_{2^n} \geq \frac{n+2}{2}$ and use it to prove that $s_{2^{n+1}} \geq \frac{n+1+2}{2}$.
We have that

$$s_{2^{n+1}} = s_{2^n} + \sum_{i=2^n+1}^{2^{n+1}} \frac{1}{i}. \tag{2}$$

For each i between $2^n + 1$ and 2^{n+1} we have that $1/i \geq 1/2^{n+1}$. Hence,

$$\sum_{i=2^n+1}^{2^{n+1}} \frac{1}{i} \geq \sum_{i=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} = \frac{2^{n+1} - 2^n}{2^{n+1}} = \frac{1}{2}. \tag{3}$$

Therefore, plugging in (3) into (2) we get

$$s_{2^{n+1}} \geq s_{2^n} + \frac{1}{2}$$

which using the induction assumption becomes

$$s_{2^{n+1}} \geq \frac{n+2}{2} + \frac{1}{2} = \frac{n+2+1}{2}$$

which is exactly the desired statement for $n + 1$.
Problem 24.2. Assume that there exists an increasing function L from $[2, \infty)$ into $(0, \infty)$ which satisfies $L(x^n) = nL(x)$. (The natural logarithm is such a function.) Determine whether the following series converge or diverge.

(a) $\sum_{n=2}^{\infty} \frac{1}{nL(n)}$

(b) $\sum_{n=2}^{\infty} \frac{1}{L(n)}$

Solution:
Both series diverge!

(a) By 2^n Test (Theorem 24.2), $\sum_{n=2}^{\infty} \frac{1}{nL(n)}$ diverges if and only if $\sum_{n=2}^{\infty} 2^n \frac{1}{2^n L(2^n)}$ does. We have that

$$2^n \frac{1}{2^n L(2^n)} = \frac{1}{nL(2)}.$$

By the contrapositive of the last part of Theorem 23.1, because $\sum_{n=1}^{\infty} 1/n$ diverges, so does $\sum_{n=1}^{\infty} 1/nL(2)$.

(b) We have that $1/L(n) > 1/nL(n)$ for $n \geq 2$, so by comparison test and part (a), $\sum_{n=2}^{\infty} \frac{1}{L(n)}$ also diverges.

Problem 24.9.

Prove that if $\{a_n\}$ is a decreasing sequence of positive numbers and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. Deduce that $\sum_{n=1}^{\infty} 1/n^s$ diverges if $0 \leq s \leq 1$.

Solution:

By 2^n Test (Theorem 24.2), the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges. Then by Theorem 22.3,

$$\lim_{n\to\infty} 2^n a_{2^n} = 0.$$

Thus, for any $\varepsilon > 0$ there exists an N such that for every $n \geq N$ we have that $|2^n a_{2^n}| < \varepsilon/2$. Given $m \geq 2^N$ let n be the smallest positive integer such that $m < 2^{n+1}$. Then

$$2^n \leq m < 2^{n+1}$$

and $n \geq N$ because $2^N \leq m < 2^{n+1}$, so $N < n+1$. Then because $m < 2^{n+1}$ and $a_m \leq a_{2^n}$ ($\{a_n\}$ is decreasing) we have that

$$|ma_m| < |2^{n+1} a_{2^n}| = 2|2^n a_{2^n}| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $|ma_m| < \varepsilon$ for every $m \geq 2^N$. Hence, $\lim_{n\to\infty} na_n = 0$.

We prove the second statement by contradiction: assume that $\sum_{n=1}^{\infty} 1/n^s$ converges for some $0 \leq s \leq 1$. Then, on one hand, by the first part of the problem, $\lim_{n\to\infty} n/n^s = 0$. However, on the other hand, by a direct computation

$$\lim_{n\to\infty} \frac{n}{n^s} = \lim_{n\to\infty} n^{1-s} = \begin{cases} \infty & \text{if } s < 1, \\ 1 & \text{if } s = 1. \end{cases}$$

Thus, our assumption leads to a contradiction.

Problem 25.4.

Give an example of a sequence $\{a_n\}$ of positive numbers such that $\lim_{n\to\infty} a_n = 0$, but the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ diverges.

Solution: The idea is to construct a sequence the sum of whose even terms diverges and the sum of whose odd terms converges. For example,

$$a_n := \begin{cases} \frac{2}{n} & \text{if } n \text{ is even}, \\ \frac{1}{2^{(n+1)/2}} & \text{if } n \text{ is odd}. \end{cases}$$

i.e.

$$\{a_n\} = \frac{1}{1}, \frac{1}{2^1}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{3}, \frac{1}{2^3}, \frac{1}{4}, \frac{1}{2^4}, \ldots$$

Let $A_n := \sum_{i=1}^{n} (-1)^{n+1} a_n$ be the partial sums of $\{a_n\}$. We show the following identity for the even index partial sums by induction on the index $2n$:

$$A_{2n} = -\sum_{i=1}^{n} \frac{1}{i} + \left(1 - \frac{1}{2^n}\right).$$

Induction basis: for $n = 1$ we have that

$$A_2 = a_1 - a_2 = -\frac{1}{2^{(1+1)/2}} + \frac{2}{2} = 1 + \left(1 - \frac{1}{2}\right).$$

Induction step: assume $A_{2n} = -\sum_{i=1}^{n} \frac{1}{i} + \left(1 - \frac{1}{2^n}\right)$. Then

$$\begin{aligned} A_{2(n+1)} = A_{2n} + a_{2n+1} - a_{2n+2} &= A_{2n} + \frac{1}{2^{n+1}} - \frac{1}{n+1} \\ &= -\left(\sum_{i=1}^{n} \frac{1}{i} + \frac{1}{n+1}\right) + \left(1 - \frac{1}{2^n} + \frac{1}{2^{n+1}}\right) \\ &= -\sum_{i=1}^{n+1} \frac{1}{i} + \left(1 - \frac{1}{2^{n+1}}\right). \end{aligned}$$

We have that $-\sum_{i=1}^{n} \frac{1}{i}$ diverges and $\left(1 - \frac{1}{2^n}\right)$ converges to 1, hence their sum A_{2n} diverges.

Problem 26.7.

Let $\{a_n\}$ be a sequence of nonzero numbers. Prove that if

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

then $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Solution:

We emulate the proof of the Ratio Test (Theorem 26.6).

(a) By Theorem 21.1

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} (\sup\{\left|\frac{a_{k+1}}{a_k}\right| \mid k \geq n\}),$$

so since $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ we can choose N such that

$$\sup\{\left|\frac{a_{n+1}}{a_n}\right| \mid n \geq N\} = M < 1.$$

Then for every $n \geq N$ we have that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq M,$$

so by induction on k we have that

$$|a_{N+k}| < M^k |a_N|.$$

Then, because the geometric series $\sum_{n=1}^{\infty} M^n |a_N|$ converges absolutely, by comparison test so does $\sum_{n=1}^{\infty} a_n$.

(b) As in the previous part, by Theorem 21.1

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}(\inf\{\left|\frac{a_{k+1}}{a_k}\right| \mid k \geq n\}),$$

, so we can choose N such that

$$\inf\{\left|\frac{a_{n+1}}{a_n}\right| \mid n \geq N\} = M > 1.$$

Then by induction on k we have that

$$|a_{N+k}| > M^k|a_N|.$$

Then because $|a_N|$ is nonzero, we have that

$$\lim_{n\to\infty} M^n|a_N| = \infty,$$

so the sequence $\{a_n\}$ is unbounded. Hence, by the contrapositive of Theorem 22.3 $\sum_{n=1}^{\infty} a_n$ diverges: if $\sum_{n=1}^{\infty} a_n$ were to converge, $\lim_{n\to\infty} a_n$ would equal zero, so $\{a_n\}$ would be bounded.

Problem 27.2.
Suppose the power series $\sum_{n=0}^{\infty} a_n(x-t)^n$ has radius of convergence R. Let p be an integer. Prove that the power series $\sum_{n=0}^{\infty} a_n(x-t)^n$ has the same radius of convergence R.
Solution:
Note that

$$|n^p a_n|^{1/n} = n^{p/n} \cdot |a_n|^{1/n}.$$

By definition of the radius of convergence, it suffices to prove that if

$$\limsup_{n\to\infty} |a_n|^{1/n} = L$$

then

$$\limsup_{n\to\infty} n^{p/n} \cdot |a_n|^{1/n} = L$$

where L is either an non-negative real number or ∞.
If $L = \infty$ then $|a_n|^{1/n}$ is unbounded above. For $n \geq 1$ we have that $n^{p/n} \geq 1$, so

$$n^{p/n} \cdot |a_n|^{1/n} \geq |a_n|^{1/n}.$$

Therefore, $n^{p/n} \cdot |a_n|^{1/n}$ is not bounded above, so $\limsup_{n\to\infty} n^{p/n} \cdot |a_n|^{1/n} = \infty$.
Assume that L is a real number. By Theorem 16.7

$$\lim_{n\to\infty} n^{1/n} = 1,$$

so because $n^{p/n} = (n^{1/n})^p$ by Corollary 12.7,

$$\lim_{n\to\infty} n^{p/n} = 1.$$

By Theorem 20.8
$$\limsup_{n\to\infty} n^{p/n}\cdot |a_n|^{1/n} = 1\cdot L = L,$$
as desired.

Problem 28.1.

Prove that the series

(a) $1+\frac{1}{\sqrt{2}}-\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{6}}+\cdots$

(b) $\sum_{n=1}^{\infty}(-1)^n n^{(1-n)/n}$

converge conditionally.

Solution:

(a) Let $\{a_n\}$ be the sequence $1, 1, -2, 1, 1, -2, \ldots$ and $\{b_n\}$ be the sequence
$$b_n = \frac{1}{\sqrt{n}}.$$
Then the sequence in question is $\{a_n b_n\}$. To show convergence using Dirichlet's Test (Theorem 28.2) it suffices to show that the sequence of partial sums $\{A_n\}$ of the series $\sum_{n=1}^{\infty} a_n$ is bounded and $\{b_n\}$ is a decreasing sequence with limit 0.

The statement about $\{b_n\}$ follow from the fact that the sequence $\{\sqrt{n}\}$ is increasing with limit ∞.

We show that the sequence $\{A_n\}$ of partial sums of $\sum_{a=1}^{\infty} i_a$s periodic with period 3: $A_{3n+1}=1$, $A_{3n+2}=2$, $A_{3n+3}=0$ by induction on n. In particular, it follows that $\{A_n\}$ is bounded below by 0 and above by 2.

The induction base follows immediately from computing the first three partial sums. In the induction step we assume that $A_{3n+3}=0$ and then get
$$\begin{aligned} A_{3n+4} &= A_{3n+3}+1 = 1\\ A_{3n+5} &= A_{3n+4}+1 = 2\\ A_{3n+6} &= A_{3n+5}-2 = 0. \end{aligned}$$
We are left to prove that $\sum_{n=1}^{\infty}|a_n b_n|$ diverges. We can do it by comparison test with $\sum_{n=1}^{\infty} b_n$ (using $|a_n b_n| \geq b_n$) which diverges by the second part of Problem 24.9 with $s = 1/2$.

(b) We will use the Alternating Series Test (Corollary 28.4) to show convergence. To use the test it suffices to show that the sequence $\{n^{(1-n)/n}\}$ is decreasing and converges to 0.

We have that
$$n^{(1-n)/n} = n^{1/n}\cdot\frac{1}{n}.$$

We know that the sequence $\{n^{1/n}\}$ is decreasing and converges to 1 (Theorem 16.7) and the sequence $\{1/n\}$ is also decreasing and converges to 0. Hence, their product $\{n^{(1-n)/n}\}$ is also decreasing and by Theorem 12.7 converges to $1 \cdot 0 = 0$.

We are left to prove that $\sum_{n=1}^{\infty} n^{(1-n)/n}$ diverges. Since the sequence $\{n^{1/n}\}$ is decreasing and converges to 1, we have that $n^{1/n}$, so

$$n^{(1-n)/n} \geq \frac{1}{n}.$$

Thus, by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} n^{(1-n)/n}$ diverges.

Problem 19.2. Let $\{a_n\}$ be a sequence. Prove that $\{a_n\}$ is a Cauchy sequence if and only if for every $\varepsilon > 0$, there exists an N such that for every $n \geq N$ $|a_n - a_N| < \varepsilon$.

Solution: Assume $\{a_n\}$ is Cauchy. Then given any $\varepsilon > 0$ there exists N such that for every $n, m \geq N$ we have that $|a_m - a_n| < \varepsilon$. In particular, we may set $m = N$ and get the desired statement.

Conversely, assume that for every $\varepsilon > 0$, there exists an N such that for every $n \geq N$ $|a_n - a_N| < \varepsilon$.

Fix an arbitrary $\varepsilon > 0$ and choose an N such that for every $n \geq N$ $|a_n - a_N| < \varepsilon/2$. Then for every $n, m \geq N$ by triangle inequality

$$|a_m - a_n| \leq |a_m - a_N| + |a_n - a_N| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\{a_n\}$ is Cauchy.

Problem 1. **More general sums:** Let $E \subset \mathbb{R}$ be any set of *positive* real numbers. Let $\mathcal{F} \subset \mathcal{P}(E)$ be the set of finite subsets of E (recall that $\mathcal{P}(E)$, the *power set* of E, is the set of all subsets), and define

$$\sum_{x \in E} x := \sup_{f \in \mathcal{F}} s_F = \sup\{s_F \mid F \in \mathcal{F}\}, \tag{4}$$

where $s_F = \sum_{f \in F} f$ is the usual sum of the elements of the finite subset $F \subset E$.

(a) Show that $\sum_{x \in E} x < \infty$ only if E is countable.

(b) Show that if E is countably infinite and $\{x_n\}$ is an *enumeration* of E (namely, $x_i = f(i)$ for $f : \mathbb{N} \xrightarrow{\cong} E$ a bijection), then

$$\sum_{x \in E} x = \sum_{i=1}^{\infty} x_i \tag{5}$$

Solution:

(a) Assume E is uncountable. We write E as a disjoint union of sets E_n indexed by a non-negative integer n:

$$E_1 = E \cap (1, \infty) \qquad \text{and} \qquad E_n = E \cap \left(\frac{1}{n}, \frac{1}{n-1}\right] \text{ for } n > 1.$$

Since E is uncountable, E_n is uncountable for at least one of n's (since a countable union of countable sets is countable, see Theorem 9.5).

Fix such an n, for which E_n is uncountable. In particular, E_n is infinite. For every positive integer m we can choose a finite subset F_m of E_n with m elements. Each element of f of F_m satisfies

$$f > \frac{1}{n}$$

being also an element of E_n. Hence,

$$\sum_{f \in F} f > \frac{m}{n}.$$

Since m can be any positive integer, the set $\{s_{F_m} \mid m \in \mathbb{N}\}$ is unbounded above, so its superset $\{s_F \mid F \in \mathcal{F}\}$ is also unbounded above and hence

$$\sum_{x \in E} x = \infty.$$

(b)

- Firstly we show that $\sum_{x \in E} x \leq \sum_{i=1}^{\infty} x_i$. Since $\sum_{x \in E} x$ is the lowest upper bound of $\{s_F \mid F \in \mathcal{F}\}$ it suffices to show that $\sum_{i=1}^{\infty} x_i$ is an upper bound of $\{s_F \mid F \in \mathcal{F}\}$, i.e. that

  $$s_F \leq \sum_{i=1}^{\infty} x_i, \ \forall F \in \mathcal{F}.$$

  Let F be an arbitrary finite subset of E. By assumption we can write F as

  $$F = \{x_i \mid i \in I\}$$

  for some finite subset I of $\mathbb{N}$. Let n be the largest element of I. Then $F \subset \{x_1, \ldots, x_n\}$, so

  $$s_F \leq \sum_{i=1}^{n} x_i.$$

  Since each x_i is positive, the sequence of partial sums $\{\sum_{i=1}^{n} x_i\}$ is increasing, so each partial sum is smaller or equal to the limit:

  $$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{\infty} x_i.$$

  Thus, $s_F \leq \sum_{=1}^{\infty} i x_i$ as desired.
- Secondly we show $\sum_{x \in E} x \geq \sum_{i=1}^{\infty} x_i$. Since $\sum_{i=1}^{\infty} x_i$ is the limit of the partial sums $\{\sum_{i=1}^{n} x_i\}$, it suffices to show that $\sum_{x \in E} \geq \sum_{i=1}^{n} x_i$ for every n. Since $F_n := \{x_1, \ldots, x_n\}$ is a finite subset of E, $s_{F_n} = \sum_{i=1}^{n} x_i$ and

  $$s_{F_n} \leq \sum_{x \in E}$$

  by definition of $\sum_{x \in E}$.

Problem 2. **Decimal (and base p) expansions:** Let $p \in \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\}$ and let x be a real number with $0 < x < 1$.

(a) Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n} \tag{6}$$

(b) Moreover, show that such a sequence $\{a_n\}$ is unique except when $x = \frac{q}{p^n}$ for another integer q; in this case, show that there are exactly two such sequences.

(c) Conversely, show that if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series (6) converges to a real number x with $0 \leq x \leq 1$.

(If $p = 10$, this $\{a_n\}$ is called the *decimal expansion* of x and gives a representation of x more familiar with from earlier math classes "$x = 0.a_1a_2a_3a_4$". If $p = 2$, this sequence is called the *binary expansion*, also mentioned in class.)

(d) Finally, consider the case $p = 2$. Let $S_{0,1}$ denote the set of *binary sequences*, by definition the set of all sequences $\{a_n\}$ where each $a_i \in \{0, 1\}$ (recall we discussed this set in class). Show using the previous two parts that there is a bijection $S_{0,1}\backslash C \cong (0, 1)$, where $C \subset S_{0,1}$ is some countable subset. Conclude that the uncountability of $S_{0,1}$ (proven in class) implies the uncountability of $\mathbb{R}$, $(0, 1)$ or any non-empty interval (a, b).

Solution:

(a) We will inductively construct a sequence $\{a_n\}$ (starting with $a_0 = 0$) such that

$$\sum_{k=0}^{n-1} \frac{a_k}{p^k} + \frac{a_n}{p^n} \leq x \leq \sum_{k=0}^{n-1} \frac{a_k}{p^k} + \frac{a_n + 1}{p^n}. \tag{7}$$

Induction basis: by assumption $0 \leq x \leq 1$, so $a_0 = 0$ satisfies (7).

Induction step: assume (7) holds for n and try to find a_{n+1} such that (7) holds for $n + 1$. Let

$$y := \left(x - \sum_{k=0}^{n} \frac{a_k}{p^k}\right) \cdot p^n.$$

By assumption (7) we have that $0 \leq y \leq 1$. Let a_{n+1} be the smallest non-negative integer such that

$$a_{n+1} + 1 \geq py. \tag{8}$$

Since $y \leq 1$, we have $(p - 1) + 1 \geq py$, so $a_{n+1} \leq p - 1$. Also, because a_{n+1} was defined to the the *smallest* non-negative integer satisfying (8), either $a_{n+1} = 0$ or $(a_{n+1} - 1) + 1 < py$. In either case,

$$a_{n+1} \leq py \leq a_{n+1} + 1.$$

After plugging in the expression for y in into the equation above and a couple of algebraic manipulations we get

$$\sum_{k=0}^{n} \frac{a_k}{p^k} + \frac{a_{n+1}}{p^{n+1}} \leq x \leq \sum_{k=0}^{n} \frac{a_k}{p^k} + \frac{a_{n+1}+1}{p^{n+1}}$$

which is exactly what we needed to prove in the induction step. Induction complete.

Next we will show that the constructed sequence $\{i_n\}$ndeed satisfies (6). By construction we have

$$0 \leq x - \sum_{k=0}^{n} \frac{a_k}{p^k} \leq \frac{1}{p^n}.$$

Hence, by Squeeze Theorem the sequence $\left\{x - \sum_{k=0}^{n} \frac{a_k}{p^k}\right\}$ converges to 0. Therefore, the sequence of partial sums $\sum_{k=0}^{n} \frac{a_k}{p^k}$ converges to x as desired.

(b)

Lemma 2. *If*

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{b_n}{p^n}$$

for two distinct *sequences $\{a_n\}$ and $\{b_n\}$ such that a_n and b_n are integers between 0 and $p-1$ then there exists a non-negative integer N such that (up to switching the sequences $\{a_n\}$ and $\{b_n\}$)*

- $a_n = b_n$ *for* $n < N$,
- $a_N = b_N + 1$,
- $a_n = 0$ *and* $b_n = p - 1$ *for* $n > N$.

Assuming the lemma, if a number x has at lest two distinct base p expansions $\{a_n\}$ and $\{b_n\}$ then

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n} = \sum_{n=1}^{N} \frac{a_n}{p^n} = \frac{q}{p^N}$$

with $q = \sum_{n=1}^{N} a_n P^{N-n}$ an integer. Conversely, we can prove that any rational number of the form q/p^N with q an integer between 1 and $p^N - 1$ has exactly two base p expansions. We start by assuming that q is not divisible by p (otherwise keep dividing both q and p^N by p until q is not divisible by p). Let $q_{N-1}q_{N-2}\dots q_0$ be a base p expansion of the integer q (where we add enough zeros at the beginning so that there are exactly N digits):

$$q = \sum_{n=0}^{N} q_n p^n.$$

Since q is not divisible by p, $q_0 > 0$.

Then q/p^N has the following two base p expansions: $\{q_{N-1}, q_{N-2}, \dots, q_1, q_0, 0, 0, 0, \dots\}$ and $\{q_{N-1}, q_{N-2}, \dots, q_1, q_0 - 1, p-1, p-1, p-1, \dots\}$, and it cannot have more than two expansions by the lemma.

Proof of Lemma 2. Let N be the minimal integer such that $a_N \neq b_N$ (it exists because we assumed $\{a_n\}$ and $\{b_n\}$ to be distinct). Assume $a_N > b_N$ (otherwise switch $\{a_n\}$ and $\{b_n\}$). Then $a_n = b_n$ for every $n < N$ and $a_N \geq b_N$ so

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a_n}{p^n} &\geq \sum_{n=1}^{N-1} \frac{a_n}{p^n} + \frac{a_N}{p^N} \\ &= \sum_{n=1}^{N-1} \frac{b_n}{p^n} + \frac{a_N}{p^N} \\ &\geq \sum_{n=1}^{N-1} \frac{b_n}{p^n} + \frac{b_N + 1}{p^N}\end{aligned}$$

with the equalities holding if and only if $a_n = 0$ for all $n > N$ and $a_N = b_N + 1$. We also have that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{b_n}{p^n} &\leq \sum_{n=1}^{N-1} \frac{b_n}{p^n} + \frac{b_N}{p^N} + \sum_{n=N+1}^{\infty} \frac{p-1}{p^n} \\ &= \sum_{n=1}^{N-1} \frac{b_n}{p^n} + \frac{b_N}{p^N} + \frac{1}{p^N}.\end{aligned}$$

with the equality holding if and only $b_n = p - 1$ for every $n \geq N$. □

(c) Since $0 \leq a_n \leq p - 1$ by the Comparison test it suffices to show that the series

$$\sum_{n=1}^{\infty} \frac{p-1}{p^n}$$

converges. The latter series is a geometric series with ratio $\frac{1}{p} < 1$, hence it converges to

$$\frac{(p-1)/p}{1 - 1/p} = 1.$$

(Theorem 22.4i). Thus, by Comparison Test (Theorem 26.3i) $\sum_{n=1}^{\infty} \frac{p-1}{p^n}$ converges and

$$0 \leq \sum_{n=1}^{\infty} \frac{p-1}{p^n} \leq 1.$$

(d) In part (b) we showed that all numbers other than those of the form $\frac{q}{2^n}$ have exactly one binary expansion and those of the form $\frac{q}{2^n}$ have exactly two: one ending in infinitely many zeros and one ending in infinitely many ones. If we prohibit binary expansions ending in all zeros we get a bijection between $(0, 1)$ and the "allowed" binary expansions.

More precisely, let C is the subset of $S_{0,1}$ consisting of sequences $\{a_n\}$ for which there exists an N such that for every $n \geq N$, $a_n = 0$. The formula (6) defines a bijection from $S_{0,1} \backslash C$ to $(0, 1)$.

To show that C is countable, note that C is equivalent to the set of numbers in $(0, 1)$ that can be written in the form $\frac{q}{2^n}$, and the latter set is a countable union of finite set: there are countably many choices of n and for each n there are finitely many choices of q.

We know from class that $S_{0,1}$ is uncountable (being equivalent to the power set $\mathcal{P}(\mathbb{N})$). Hence, $S_{0,1} \setminus C$ is also uncountable (because if it were countable then $S_{0,1}$ would be a union of two countable sets and hence also countable).