Math 171 Homework 3

Due Friday April 22, 2016 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Alex Zamorzaev, in his office, 380-380M (either hand your solutions directly to him or leave the solutions under his door).

Book problems: Solve Johnsonbaugh and Pfaffenberger, problems 33.3, 35.5, 35.6, 35.7, 36.8, 36.11, 37.4, 37.10, 40.10. Also solve the following two problems.

First, a definition which will be helpful for both problems:

DEFINITION 1. *A pair of metric spaces* (A, d_A) *and* (B, d_B) *are* isometric *if there is a bijection of sets* $f : A \xrightarrow{\cong} B$ *which preserves distances, meaning that* $d_B(f(x), f(y)) = d_A(x, y)$ *for all* x, y*. The distance-preserving bijection* f *is called an* isometry.

1. Product metrics, continuity, and composition. If (M_1, d_1) and (M_2, d_2) are metric spaces, then one can define a distance function d on the Cartesian product $M_1 \times M_2$ by

$$d((m,n),(m',n')) = d_1(m,m') + d_2(n,n') \tag{1}$$

(a) Show that d defines a metric on $M_1 \times M_2$, called the *(standard) product metric.*

(b) In fact, d is not the only metric on $M_1 \times M_2$. For instance, show that

$$d_2((m,n),(m',n')) = \sqrt{d_1(m,m')^2 + d_2(n,n')^2} \tag{2}$$

defines another metric on $M_1 \times M_2$, called the ℓ^2 *product metric.*
Then, also show that there is an isometry between $\mathbb{R}^n$ with its Euclidean metric and $\mathbb{R}^k \times \mathbb{R}^{n-k}$ where each of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ are equipped with their Euclidean metrics and the product is equipped with the ℓ^2 product metric.

(c) Usually, unless otherwise specified, we will think of the product $M_1 \times M_2$ as a metric space with respect to the standard product metric. Show that if N is a metric space and $f_i : N \to M_i$ are continuous maps for $i = 1, 2$, then the map $(f_1, f_2) : N \to M_1 \times M_2$ is continuous.

Now, show that the identity map $(M_1 \times M_2, d) \to (M_1 \times M_2, d_{\ell^2})$ is a continuous map from $M_1 \times M_2$ with its standard product metric to $M_1 \times M_2$ with its ℓ^2 product metric.

Conclude from Corollary 40.6 of the book (which says that the composition of continuous functions is continuous) that if f_1 and f_2 are continuous, then $(f_1, f_2) : N \to M_1 \times M_2$, where now $M_1 \times M_2$ is equipped with the ℓ^2 product metric, is continuous too.

(d) Prove that the following functions from $\mathbb{R}^2 \to \mathbb{R}$ (where $\mathbb{R}^2$ is equipped with its Euclidean metric) are continuous: $(x, y) \mapsto xy$, $(x, y) \mapsto x + y$. Deduce a new proof of Theorem 40.4 parts (ii) and (v) from this, part (c), and the fact that compositions of continuous functions are continuous (Corollary 40.6). Finally, indicate (but do not prove) how the remaining parts of Theorems 40.4 are similarly consequences of the continuity of some basic real-valued functions on $\mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$, along with the part (c) and Corollary 40.6.

2. Pseudometrics and equivalence relations. Recall that an *equivalence relation* on a set X is any binary relation[1] on X, denoted $\sim$, which satisfies the following properties:

- (*reflexive property*) $x \sim x$ for all elements $x \in X$;
- (*symmetric property*) $x \sim y$ implies $y \sim x$, for any pair of elements $x, y \in X$;
- (*transitive property*) If $x \sim y$ and $y \sim z$, then $x \sim z$, for any triple of elements $x, y, z \in X$.

One simple example of an equivalence relation is *equality*: it is clear that $x = x$, $x = y$ if $y = x$, and if $x = y$ and $y = z$ then $x = z$.

Now, let M be a set. A function $d : M \times M \to [0, \infty)$ is called a *pseudometric* if d satisfies

(i) $d(x, x) = 0$ for any x;
(ii) $d(x, y) = d(y, x)$ for any x, y; and
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for any x, y, z.

Namely, d satisfies all of the conditions of a metric, except that $d(x, y) = 0$ need not imply $x = y$ (note that condition (i) is weaker than the usual condition for a metric). A *pseudometric space* is a pair (M, d) of a set and a pseudometric on it.

(a) Show that $\mathbb{R}^2$ with the function $d(x, y) = |y_2 - x_2|$ is a pseudometric space, but not a metric space (that is, that $d(x, y) = 0$ does not imply $x = y$).

(b) Given a pseudometric space, (M, d), consider the following binary relation on M: say $x \sim y$ if $d(x, y) = 0$. Show that $\sim$ is an equivalence relation.

(c) Recall that given a set X and an equivalence relation $\sim$ on X, one can partition X into a collection of *equivalence classes*. An equivalence class is a subset of X consisting of all elements that are similar to a given element. Any element $a \in X$ belongs to a single equivalence class, called $[a]$:

$$[a] := \{x \in X \mid x \sim a\}.$$

Two elements a and b have the same equivalence class ($\alpha = [a] = [b]$) if and only if $a \sim b$ (a and b are both called *representatives* of the equivalence class).
Show that given a pseudometric space (M, d), the function $d(x, y)$ only depends on the equivalence classes $[x]$ $[y]$ with respect to the equivalence relation $\sim$.

(d) Given a set with an equivalence relation $(X, \sim)$, the set of distinct equivalence classes, also called the *quotient* of X by $\sim$, is denoted

$$X/\sim \; := \; \{[a] \mid a \in X\}.$$

Given a pseudometric space (M, d), define $M^* := M/\sim$, where $\sim$ is the equivalence relation defined above. Define $d^* : M^* \times M^* \to [0, \infty)$ by $d^*(\alpha, \beta) = d(x, y)$ for any representatives $x \in \alpha$, $y \in \beta$. By the previous section, $d(x, y)$ only depends on the equivalence classes $[x]$, $[y]$, so d^* is well-defined.

Show that d^* gives a metric on M^*.

[1] Formally, a *binary relation* on X is a subset $R \subset X \times X$ of pairs of elements in X. One often uses a shorthand xRy, $x \sim_R y$, $x \sim y$ to indicate that $(x, y) \in R$. Examples of binary relations on $\mathbb{R}$ include the usual $<, \geq, >, \leq, =$, etc.

(e) Let's return to the example $X = (\mathbb{R}^2, d(x,y) = |y_2 - x_2|)$. Show that the induced metric space (X^*, d^*) is *isometric* to $\mathbb{R}$ with its Euclidean metric.

(f) Let $\mathcal{F}(\mathbb{R})$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$, Fixing a pair of distinct $x_0, x_1 \in \mathbb{R}$, define a function

$$d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to [0, \infty)$$

by $d(f,g) = |f(x_0) - g(x_0)| + |f(x_1) - g(x_1)|$. Show that d defines a pseudometric on $\mathcal{F}(\mathbb{R})$, but that $d(x,y) = 0$ does not imply $x = y$. Show that the resulting quotient metric space $\mathcal{F}(\mathbb{R})^*$ is isometric to $\mathbb{R}^2$ with its ℓ^1 metric.