Math 171 Homework 3

(due April 22)

Problem 33.3.
Let f be defined on $[0,1]$ by the formula

$$f = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f is continuous only at 0.
Solution:
Firstly, we show that f is *not* continuous at any non-zero x by constructing a sequence $\{x_n\}$ converging to x such that its image $\{f(x_n)\}$ does not converge to $f(x)$. We consider two cases:

- Case 1: x is rational, so $f(x) = x \neq 0$.

  In this case consider the sequence $x_n := x + \frac{1}{n\sqrt{2}}$. Each x_n is irrational, so $f(x_n) = 0$. Then $\{f(x_n\}$ converges to 0, and not to $f(x) = x$.

- Case 2: x is irrational, so $f(x) = 0$.

  In this case consider a sequence of *rational* numbers $\{x_n\}$ converging to x. We can find such a sequence by finding a rational number x_n in the interval $(x, x+1/n)$ for each n by Theorem 7.8. Then $f(x_n) = x_n$ for each n, so the sequence $\{f(x_n)\}$ converges to x and not to $f(x) = 0$.

Secondly, we show that f is continuous at 0 using Theorem 33.3. We claim that given an $\varepsilon > 0$, for any $x \in \mathbb{R}$ such that $|x| < \varepsilon$ we have $|f(x)| < \varepsilon$ (we're using $f(0) = 0$ here). There are two cases here again:

- Case 1: x is rational, so $f(x) = x$ and consequently $|f(x)| < \varepsilon$ by our assumption on x.

- Case 2: x is irrational, so $f(x) = 0$ and consequently $|f(x)| = 0 < \varepsilon$.

Problem 35.5.
Let (M, d) be a metric space and let X be a subset of M. Prove that $(X, d|_{X\times X})$ is a metric space.
Solution:
We need to check that d still satisfies the axioms of a metric (Definition 35.1) after we restrict it to $X \times X$:

(i) We know that $d(x,x) = 0$ for every $x \in M$, in particular, for every $x \in X$. Conversely, if $d(x,y) = 0$ with $x, y \in X$ then $x = y$ because $x, y \in M$ and d is a metric on M.

(ii) We know that $d(x,y) = d(y,x)$ for any $x, y \in M$, in particular, for $x, y \in X$.

(iii) We know that the triangle inequality holds for $x, y, z \in M$, so in particular it holds for $x, y, z \in X$.

Problem 35.6.
Let l^∞ denote the set of all bounded real sequences, and let c_0 denote the set of all real sequences which converge to 0.

(a) Prove that $l^1 \subset c_0 \subset l^\infty$.

(b) Prove that the containments in (a) are proper.

(c) Prove that
$$d_{l^\infty}(\{a_n\}, \{b_n\}) := \sup\{|a_n - b_n| \mid n \in \mathbb{N}\}$$
defines a metric on l^∞ (and thus by (a) together with Problem 35.5 also on c_0 and l^1)

(d) Let d_{l^1} be the metric on l^1 defined in Example 35.5. Prove that
$$d_{l^\infty}(x, y) \leq d_{l^1}(x, y)$$
for $x, y \in l^1$.

Solution:

(a) By Theorem 22.3, $l^1 \subset c_0$ and by Theorem 13.2, $c_0 \subset l^\infty$.

(b) The harmonic series $a_n = 1/n$ is an element of c_0 but not of l_1. The constant series $b_n = 1$ is an element of l_∞ but not c_0.

(c) We verify the axioms of a metric:

(i) Assume $\{a_n\} = \{b_n\}$. Then $a_n = b_n$ for every $n \in N$, so
$$d_{l^\infty}(\{a_n\}, \{a_n\}) = \sup\{0\} = 0.$$
Conversely, if $\{a_n\} \neq \{b_n\}$, then there exists an $n \in \mathbb{N}$ such that $a_n - b_n \neq 0$, so $|a_n - b_n| > 0$ for that n. Hence 0 is not an upper bound of $\{|a_n - b_n| \mid n \in \mathbb{N}\}$, so $d_{l^\infty}(\{a_n\}, \{b_n\}) > 0$ in that case.

(ii) Symmetry: given $\{a_n\}$ and $\{b_n\}$ in l^∞ we have $|a_n - b_n| = |b_n - a_n|$ for every $n \in \mathbb{N}$, so
$$\{|a_n - b_n| \mid n \in \mathbb{N}\} = \{|b_n - a_n| \mid n \in \mathbb{N}\}$$
and, therefore,
$$d_{l^\infty}(\{a_n\}, \{b_n\}) = d_{l^\infty}(\{b_n\}, \{a_n\}).$$

(iii) Given $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in l^∞, for every $n \in \mathbb{N}$ we have that
$$|a_n - b_n| \leq d_{l^\infty}(\{a_n\}, \{b_n\}) \qquad \text{and} \qquad |b_n - c_n| \leq d_{l^\infty}(\{b_n\}, \{c_n\}).$$
By triangle inequality for the absolute value
$$|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|,$$

so

$$|a_n - c_n| \le d_{l^\infty}(\{a_n\}, \{b_n\}) + d_{l^\infty}(\{b_n\}, \{c_n\})$$

for every $n \in \mathbb{N}$. Thus, $d_{l^\infty}(\{a_n\}, \{b_n\}) + d_{l^\infty}(\{b_n\}, \{c_n\})$ is an upper bound of $\{|a_n - c_n| \mid n \in \mathbb{N}\}$, so

$$d_{l^\infty}(\{a_n\}, \{b_n\}) \le d_{l^\infty}(\{a_n\}, \{b_n\}) + d_{l^\infty}(\{b_n\}, \{c_n\}),$$

as desired.

(d) For every $n \in \mathbb{N}$,

$$|x_n - y_n| \le \sum_{k=1}^{\infty} |x_k - y_k| = d_{l^1}(x, y),$$

so $d_{l^1}(x, y)$ is an upper bound on $\{|x_n - y_n| \mid n \in \mathbb{N}\}$, and hence its greater or equal to the least upper bound:

$$d_{l^\infty}(x, y) \le d_{l^1}(x, y).$$

Problem 35.7.

Let (M, d) be a metric space. Prove that

$$d'(x, y) := \frac{d(x, y)}{1 + d(x, y)} \qquad \text{and} \qquad d''(x, y) := \min\{d(x, y), 1\}$$

define metrics on M. Prove that d' and d'' are bounded by 1.

Solution:

We verify the axioms of a metric for d' and d'':

(i)

$$d'(x, x) = \frac{d(x, x)}{1 + d(x, x)} = \frac{0}{1 + 0} = 0.$$

$$d''(x, x) = \min\{d(x, x), 1\} = \min\{0, 1\} = 0.$$

Conversely, if $x \ne y$ then $d(x, y) > 0$, so

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} > 0$$

because the numerator and the denominator are positive and

$$d''(x, y) = \min\{d(x, y), 1\} > 0.$$

because both $d(x, y)$ and 1 are positive.

(ii) By assumption $d(x, y) = d(y, x)$, so

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = d'(y, x)$$

and $\{d(x, y), 1\} = \{d(y, x), 1\}$, so

$$d''(x, y) = \min\{d(x, y), 1\} = \min\{d(y, x), 1\} = d''(y, x).$$

(iii) The triangle inequality for d'. We have that

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq \frac{d(x,y)}{1+d(x,y)+d(y,z)}$$

and

$$d'(y,z) = \frac{d(y,z)}{1+d(y,z)} \geq \frac{d(y,z)}{1+d(x,y)+d(y,z)}.$$

Thus,

$$d'(x,y) + d'(y,z) \geq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \geq \frac{d(x,z)}{1+d(x,z)} = d'(x,z)$$

where the latter inequality holds because $d(x,y) + d(y,z) \geq d(x,z)$ and the function $f(t) := \frac{t}{1+t} = 1 - \frac{1}{1+t}$ is increasing in t for $t \in [0,\infty)$.

The triangle inequality for d''. We consider two cases:

Case 1: at least one of the two inequalities $d(x,y) \geq 1$ or $d(y,z) \geq 1$ holds. In this case at least one of the two equalities $d''(x,y) = 1$ or $d''(y,z) = 1$ holds, so

$$d''(x,y) + d''(y,z) \geq 1 \geq \min\{d(x,z), 1\} = d''(x,z).$$

Case 2: $d(x,y) < 1$ and $d(y,z) < 1$. Hence, $d''(x,y) = d(x,y)$ and $d''(y,z) = d(y,z)$. Therefore,

$$d''(x,y) + d''(y,z) \geq d(x,z) \geq \min\{d(x,z), 1\} = d''(x,z).$$

Problem 36.8.
Let $\{a_n\} \in l^1$ and $\{b_n\} \in l^\infty$. Prove that $\{a_n b_n\} \in l^1$.
Solution:
This is exactly the statement of Theorem 26.4(i).
Problem 36.11.
Let $\{a_n\}$ be a sequence such that $\{a_n b_n\} \in l^1$ for every sequence $\{b_n\} \in l^1$. Prove that $\{a_n\} \in l^\infty$. Show (by example) that the above statement is false l^∞ is replaced by c_0.
Solution:
We prove the contrapositive: if $\{a_n\}$ is not in l^∞ then there exists a sequence $\{b_n\}$ in l^1 such that $\{a_n b_n\} \notin l^1$.
If $\{a_n\}$ is not in l^∞, it is not bounded. Assume $\{a_n\}$ is not bounded from above (otherwise replace $\{a_n\}$ with $\{-a_n\}$). Then we can inductively construct a subsequence $\{a_{n_k}\}$ such that $a_{n_k} > k$.
Let

$$b_n := \begin{cases} \frac{1}{k^2} & \text{if } n = n_k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $0 \leq b_n \leq 1/n^2$ for every $n \in \mathbb{N}$, so by the Comparison Test, $b \in l^1$. However,

$$a_n b_n = \begin{cases} \frac{1}{k} & \text{if } n = n_k \text{ for some } k, \\ 0 & \text{otherwise,} \end{cases}$$

so the subsequence $\{\sum_{n=1}^{n_k} a_n b_n\}_{k\in\mathbb{N}} = \{\sum_{m=1}^{k} \frac{1}{k}\}_{k\in\mathbb{N}}, m$ of partial sums of $\{a_n b_n\}$ diverges. Hence, $\{a_n b_n\}$ is not an element of l^1.
The constant sequence $a_n = 1$ (tautologically) satisfies the premise of the problem and is not an element of c_0 because its limit is 1 and not 0.

Problem 37.4.

Show (by examples) that Theorem 37.2 does not generalize to l^2, c_0 or l^∞.

Solution:

Define the sequence $a^{(k)} = \{a_n^{(k)}\}$ by

$$a_n^{(k)} := \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. $a^{(k)}$ has a 1 at index k and zeros everywhere else. Then $a^{(k)}$ is an element of l^1 (and hence of c_0 and l^∞ too)
Then $\lim_{k\to\infty} a_n^{(k)} = 0$ for every $n \in \mathbb{N}$ because $a_n^{(k)} = 0$ for all $k > n$.
However, the sequence $\{a^{(k)}\}$ of sequences does not converge to the sequence $\underline{0}$ consisting of all zeros in l^∞ (and, hence, by Problem 35.6(d) it does not converge to $\underline{0}$ in l^1 either) because

$$d_{l^\infty}(a^{(k)}, \underline{0}) = \sup\{0, 1\} = 1$$

for every k.

Problem 37.10.

Let (M, d) be a metric space and let d' and d'' be defined as in Exercise 35.7. Let $\{a_n\}$ be a sequence in M and let $a \in M$. Prove that the following statements are equivalent:

(a) $\{a_n\}$ converges to a in (M, d).

(b) $\{a_n\}$ converges to a in (M, d').

(c) $\{a_n\}$ converges to a in (M, d'').

Solution:

- (a) $\Rightarrow$ (b) and (c)

  We have that $\lim_{n\to\infty} d(a_n, a) = 0$ and also

  $$0 \le d'(a_n, a) \le d(a_n, a) \qquad \text{and} \qquad 0 \le d''(a_n, a) \le d(a_n, a).$$

  Therefore, by Squeeze Theorem both $d'(a_n, a)$ and $d''(a_n, a)$ converge to 0.

- (b) $\Rightarrow$ (a)

  Given $\epsilon \in (0, 1)$, choose N such that

  $$d'(a_n, a) < \frac{\epsilon}{1+\epsilon}$$

  for every $n \ge N$. Then

  $$d(a_n, a) = \frac{1}{1 - d'(a_n, a)} - 1 < \epsilon$$

  for every $n \ge N$. Hence, $\lim_{n\to\infty} d(a_n, a) = 0$.

- (c) $\Rightarrow$ (a)

  Since $\lim_{n\to\infty} d''(a_n, a) = 0$, we can choose N such that $d''(a_n, a) < 1$ for all $n \geq N$. Using the definition of d'' we get that for all $n \geq N$, $d''(a_n, a) = d(a_n, a)$, so $\lim_{n\to\infty} d(a_n, a) = 0$.

Problem 40.10.
Let $\{a_n\} \in l^\infty$. Prove that f defined by

$$f(\{b_n\}) := \sum_{n=1}^{\infty} a_n b_n$$

is a continuous real-valued function on l^1.
Solution:
To show continuity of f we need to show that given $\{b_n\} \in l^1$ and $\varepsilon > 0$ we can find $\delta > 0$ such that for every $\{\tilde{b}_n\} \in l^1$ such that $d_{l^1}(\{b_n\}, \{\tilde{b}_n\}) < \delta$ we have that

$$|f(\{b_n\}) - f(\{\tilde{b}_n\})| < \varepsilon.$$

Assume $\{a_n\}$ is bounded by M with $M > 0$ in absolute value: $|a_n| < M$ for every $n \in \mathbb{N}$. We show that $\delta := \varepsilon/M$ works.
By Theorem 23.1 applied to series $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} a_n \tilde{b}_n$ we have that

$$f(\{b_n\}) - f(\{\tilde{b}_n\}) = \sum_{n=1}^{\infty} a_n(b_n - \tilde{b}_n).$$

Hence, by Theorem 26.2

$$|f(\{b_n\}) - f(\{\tilde{b}_n\})| \leq \sum_{n=1}^{\infty} |a_n(b_n - \tilde{b}_n)|$$

By Theorem 26.3(i) we have that

$$\sum_{n=1}^{\infty} |a_n(b_n - \tilde{b}_n)| \leq \sum_{n=1}^{\infty} M|b_n - \tilde{b}_n| = M d_{l^1}(\{b_n\}, \{\tilde{b}_n\}) < \epsilon.$$

Therefore,

$$|f(\{b_n\}) - f(\{\tilde{b}_n\})| < \epsilon,$$

as desired.
Problem 1. Product metrics, continuity and composition..
If (M_1, d_1) and (M_2, d_2) are metric spaces then one can define a distance function d on the Cartesian product $M_1 \times M_2$ by

$$d((m, n), (m', n')) := d_1(m, m') + d_2(n, n'). \tag{1}$$

(a) Show that d defines a metric on $M_1 \times M_2$ called the *(standard) product metric.*

(b) In fact, d is not the only natural metric on $M_1 \times M_2$. For instance, show that

$$d_{\ell^2}((m,n),(m',n')) := \sqrt{d_1(m,m')^2 + d_2(n,n')^2} \tag{2}$$

defines another metric on $M_1 \times M_2$, called the ℓ^2 *product metric.*

Then, also show that there is an isometry between $\mathbb{R}^n$ with its Euclidean metric and $\mathbb{R}^k \times \mathbb{R}^{n-k}$ where each of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ are equipped with their Euclidean metrics and the product is equipped with the l^2 product metric.

(c) Usually, unless otherwise specified, we will think of the product $M_1 \times M_2$ as a metric space with respect to the standard product metric. Show that if N is a metric space and $f_i : N \to M_i$ are continuous maps for $i = 1, 2$, then the map $(f_1, f_2) : N \to M_1 \times M_2$ is also continuous.

Now, show that the identity map $(M_1 \times M_2, d) \to (M_1 \times M_2, d_{\ell^2})$ is a continuous map from $M_1 \times M_2$ with its standard product metric to $M_1 \times M_2$ with its ℓ^2 product metric.

Conclude from Corollary 40.6 of the book (which says that the composition of continuous functions is continuous) that if f_1 and f_2 are continuous, then $(f_1, f_2) : N \to M_1 \times M_2$, where now $M_1 \times M_2$ is equipped with the ℓ^2 product metric, is continuous too.

(d) Prove that the following functions from $\mathbb{R}^2$ to $\mathbb{R}$ (where $\mathbb{R}^2$ is equipped with its Euclidean metric) are continuous:

- $(x, y) \mapsto xy$,
- $(x, y) \mapsto x + y$.

Deduce a new proof of Theorem 40.4 parts (ii) and (v) from this, part (c), and the fact that the compositions of continuous functions are continuous (Corollary 40.6). Finally, indicated (but do not prove) how the remaining parts of Theorem 40.4 are similarly consequences of the continuity of some basic real-valued functions on $\mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$ along with the part (c) and Corollary 40.6.

Solution:

(a) We check the axioms of a metric for d:

(i) We have $d_1(m, m) = 0$ and $d_2(n, n) = 0$, hence

$$d((m,n),(m,n)) = d_1(m,m) + d_2(n,n) = 0.$$

Conversely, if $(m, n) \neq (m', n')$ then at least one of the following non-equalities hold: $m \neq m'$ or $n \neq n'$. Hence, at least one of the following inequalities hold: $d_1(m, m') > 0$ or $d_2(n, n') > 0$. Thus,

$$d((m,n),(m',n')) = d_1(m,m') + d_2(n,n') > 0.$$

(ii) Symmetry:

$$d((m,n),(m',n')) = d_1(m,m') + d_2(n,n') = d_1(m',m) + d_2(n',n) = d((m',n'),(m,n)).$$

(iii) Triangle inequality: we know that

$$d_1(m,m'') \leq d_1(m,m') + d_1(m',m'') \qquad \text{and} \qquad d_2(n,n'') \leq d_2(n,n') + d_2(n',n'').$$

Hence

$$\begin{aligned} d((m,n),(m'',n'')) &= d_1(m,m'') + d_2(n,n'') \\ &\leq (d_1(m,m') + d_1(m',m'')) + (d_2(n,n') + d_2(n',n'')) \\ &= (d_1(m,m') + d_2(n,n')) + (d_1(m',m'') + d_2(n',n'')) \\ &= d((m,n),(m',n')) + d((m',n'),(m'',n'')). \end{aligned}$$

(b) We check the axioms of a metric for d_{ℓ^2} in the same manner as in the previous part:

(i) We have $d_1(m,m) = 0$ and $d_2(n,n) = 0$, hence

$$d_{\ell^2}((m,n),(m,n)) = \sqrt{d_1(m,m)^2 + d_2(n,n)^2} = 0.$$

Conversely, if $(m,n) \neq (m',n')$ then at least one of the following non-equalities hold: $m \neq m'$ or $n \neq n'$. Hence, at least one of the following inequalities hold: $d_1(m,m') > 0$ or $d_2(n,n') > 0$. Thus,

$$d_{\ell^2}((m,n),(m',n')) = \sqrt{d_1(m,m')^2 + d_2(n,n')^2} > 0.$$

(ii) Symmetry:

$$\begin{aligned} d_\ell^2((m,n),(m',n')) &= \sqrt{d_1(m,m')^2 + d_2(n,n')^2} \\ = \sqrt{d_1(m',m)^2 + d_2(n',n)^2} &= d_\ell^2((m',n'),(m,n)). \end{aligned}$$

(iii) Triangle inequality: as before, we know that

$$d_1(m,m'') \leq d_1(m,m') + d_1(m',m'') \qquad \text{and} \qquad d_2(n,n'') \leq d_2(n,n') + d_2(n',n'').$$

Hence,

$$\begin{aligned} d((m,n),(m'',n'')) &= \sqrt{d_1(m,m'')^2 + d_2(n,n'')^2} \\ &\leq \sqrt{(d_1(m,m') + d_1(m',m''))^2 + (d_2(n,n') + d_2(n',n''))^2} \\ &\leq \sqrt{d_1(m,m')^2 + d_2(n,n')^2} + \sqrt{d_1(m',m'')^2 + d_2(n',n'')^2} \\ &= d((m,n),(m',n')) + d((m',n'),(m'',n'')), \end{aligned}$$

where the latter inequality holds by Lemma 1.

Lemma 1. *Any real numbers a_1, a_2, b_1, b_2 satisfy*

$$\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \leq \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2}$$

Proof. The desired inequality is equivalent to the following triangle inequality in $\mathbb{R}^2$:

$$d_{\mathbb{R}^2}((0,0),(a_1+a_2,b_1+b_2)) \leq d_{\mathbb{R}^2}((0,0),(a_1,b_1)) + d_{\mathbb{R}^2}((a_1,b_1),(a_1+a_2,b_1+b_2)).$$

$\square$

We can give an isometry $f : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ via concatenation of vectors:

$$f((a_1,\ldots,a_k),(b_1,\ldots,b_{n-k})) = (a_1,\ldots,a_k,b_1,\ldots,b_{n-k}).$$

It is an isometry because, given $a, a' \in \mathbb{R}^k$ and $b, b' \in \mathbb{R}^{n-k}$ we have that

$$\begin{aligned} d_{\mathbb{R}^k\times\mathbb{R}^{n-k}}((a,b),(a',b')) &= \sqrt{d_{\mathbb{R}^k}(a,a')^2 + d_{\mathbb{R}^{n-k}}(b,b')^2} \\ &= \sqrt{\sum_{j=1}^{k} |a_j - a'_j|^2 + \sum_{j=1}^{n-k} |b_j - b'_j|^2} \\ &= d_{\mathbb{R}^n}(f(a,b), f(a',b')). \end{aligned}$$

(c) Since f_1, f_2 are continuous, then given $x \in N$ and $\varepsilon > 0$ we can choose $\delta_1, \delta_2 > 0$ such that for every $y \in N$ such that $d_N(x,y) < \delta_i$ we have $d_{M_i}(f_i(x), f_i(y)) < \varepsilon/2$ (with $i = 1,2$). Let $\delta := \min(\delta_1, \delta_2)$. Then for every $y \in y$ such that $d_N(x,y) < \delta$ we have that

$$\begin{aligned} d_{M_1\times M_2}((f_1(x),f_2(x)),(f_1(y),f_2(y)) &= d_{M_1}(f_1(x),f_1(y)) + d_{M_2}(f_2(x),f_2(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, (f_1, f_2) is continuous.

To show that the identity $(M_1 \times M_2, d) \to (M_1 \times M_2, d_{\ell^2})$ is continuous we show

$$d_{\ell^2}(x,y) \leq d(x,y)$$

for every $x, y \in M_1 \times M_2$. Indeed, assuming the inequality above, given $\varepsilon > 0$ and $x \in M_1 \times M_2$ for every $y \in M_1 \times M_2$ such that $d(x,y) < \varepsilon$ we also have $d_{\ell^2}(x,y) < \varepsilon$.

The desired inequality is equivalent to the following two inequalities

$$\begin{aligned} &\sqrt{d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2} \leq d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2 \\ \iff &d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2 \leq d_1(x_1,y_1)^2 + 2d_1(x_1,y_1)\cdot d_2(x_2,y_2) + d_2(x_2,y_2)^2, \end{aligned}$$

with the latter one is being obviously true.

We have that $(f_1, f_2) : (N, d_N) \to (M_1 \times M_2, d_{\ell^2})$ is a composition of two continuous continuous functions: $(f_1, f_2) : (N, d_N) \to (M_1 \times M_2, d)$ and identity $(M_1 \times M_2, d) \to (M_1 \times M_2, d_{\ell^2})$, hence it is continuous by Corollary 40.6.

(d) By Theorem 40.2, to show continuity of $(x, y) \mapsto xy$ and $(x, y) \mapsto x + y$ it suffices to show that for any sequence $\{(x_n, y_n)\}$ points in $\mathbb{R}^2$ converging to some $(x, y) \in \mathbb{R}^2$, the sequences $\{x_n y_n\}$ and $\{x_n + y_n\}$ converge to xy and $x + y$, respectively.

By Theorem 37.2, $\{x_n\}$ and $\{y_n\}$ converge to x and y, respectively. Hence, the desired conclusion follows by Theorem 12.6 and 12.2, respectively.

The remaining parts of Theorem 40.4 follow from the continuity of

(i) $x \mapsto |x| : \mathbb{R} \to \mathbb{R}$,

(iii) $x \mapsto cx : \mathbb{R} \to \mathbb{R}$,

(iv) $(x, y) \mapsto x - y : \mathbb{R}^2 \to \mathbb{R}$,

(vi) $(x, y) \mapsto x/y : \mathbb{R} \times \mathbb{R} \backslash \{0\} \to \mathbb{R}$.

Problem 2. Pseudometrics and equivalence relations..

Recall that an *equivalence relation* on a set X is any binary relation on X, denoted $\sim$, which satisfies the following properties:

- *(reflexive property)* $x \sim x$, $\forall x \in X$;
- *(symmetric property)* $x \sim y$, implies $y \sim x$, $\forall x, y \in X$;
- *(transitive property)* If $x \sim y$ and $y \sim z$ then $x \sim z$, $\forall x, y, z \in X$.

One simple example of an equivalence relation is *equality*: it is clear that $x = x$, $x = y$ if $y = x$ and if $x = y$ and $y = z$ then $x = z$.
Now, let M be a set. A function $d : M \times M \to [0, \infty)$ is called a *pseudometric* if d satisfies

(i) $d(x, x) = 0$, $\forall x \in M$;

(ii) $d(x, y) = d(y, x)$, $\forall x, y \in M$ and

(iii) $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in M$.

Namely, d satisfies all of the conditions of a metric except that $d(x, y) = 0$ need not imply $x = y$ (note that condition (i) is weaker than the respective condition for a metric). A *pseudometric space* is a pair (M, d) of a set and a pseudometric on it.

(a) Show that $\mathbb{R}^2$ with the function $d(x, y) = |y_2 - x_2|$ is a pseudometric space but not a metric space (that is, $d(x, y) = 0$ does not imply $x = y$).

(b) Given a pseudometric space (M, d), consider the following binary relation on M: say $x \sim y$ if $d(x, y) = 0$. Show that $\sim$ is an equivalence relation.

(c) Recall that given a set X and an equivalence relation $\sim$ on X one can partition X into a collection of *equivalence classes*. An equivalence class is a subset of X consisting of all elements that are similar to a given element. Any element $a \in X$ belongs to a single equivalence class called $[a]$:

$$[a] := \{x \in X \mid x \sim a\}.$$

Two elements a and b have the same equivalence class (i.e. $[a] = [b]$) if and only if $a \sim b$. In that case a and b are different *representatives* of the same equivalence class.

Show that given a pseudometric space (M, d), the function $d(x, y)$ only depends on the equivalence classes $[x], [y]$ with respect to the equivalence relation $\sim$.

(d) Given a set with an equivalence relation $(X, \sim)$ we can define the *quotient* of X by $\sim$ as the set of distinct equivalence classes:

$$X/ \sim := \{[a] \mid a \in X\}.$$

Given a pseudometric space (M, d), define $M^* := M/ \sim$ where $\sim$ is the equivalence relation defined above. Define $d^* : M^* \times M^* \to [0, \infty)$ by $d^*(\alpha, \beta) := d(x, y)$ for any representatives $x \in \alpha, y \in \beta$. By the previous section, $d(x, y)$ only depends on the equivalence classes $[x], [y]$, do d^* is well-defined.

Show that d^* gives a metric on M^*.

(e) Let's return to the example $X = (\mathbb{R}^2, d(x, y) = |y_2 - x_2|)$. Show that the induced metric space (X^*, d^*) is *isometric* to $\mathbb{R}$ with its Euclidean metric.

(f) Let $\mathcal{F}(\mathbb{R})$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$. Fixing a pair of distinct $x_0, x_1 \in \mathbb{R}$, define a function

$$d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to [0, \infty)$$

by

$$d(f, g) = |f(x_0) - g(x_0)| + |f(x_1) - g(x_1)|.$$

Show that d defines a pseudometric on $\mathcal{F}(\mathbb{R})$ but that $d(x, y) = 0$ does not imply $x = y$. Show that the resulting quotient metric space $\mathcal{F}(\mathbb{R})^*$ is isometric to $\mathbb{R}^2$ with its ℓ^1 metric.

Solution:

(a) We check the axioms of a pseudometric space for d:

(i) $d(x, x) = |x_2 - x_2| = 0$.

(ii) $d(x, y) = |y_2 - x_2| = |x_2 - y_2| = d(y, x)$

(iii) $d(x, z) = |z_2 - x_2| \le |y_2 - x_2| + |z_2 - y_2| = d(x, y) + d(y, z)$.

However, d is not a metric because $d((0, 0), (1, 0)) = 0$.

(b) We have that $x \sim y$ if and only if $x_2 = y_2$. We check the axioms of an equivalence for this relation:

- Reflexivity: $x \sim x$ because $d(x, x) = 0$.
- Symmetry: if $x \sim y$ then $d(x, y) = 0$. By symmetry of the metric $d(y, x) = 0$, hence $y \sim x$.

- Transitivity: if $x \sim y$ and $y \sim z$ then $d(x,y) = 0$ and $d(y,z) = 0$, so by triangle inequality
$$d(x,z) \leq d(x,y) + d(y,z) = 0.$$
Since $d(x,z)$ cannot be negative, $d(x,z) = 0$, so $x \sim z$

(c) It suffices to show that $d(x,y) = d(x,y')$ whenever $y \sim y'$. Indeed, in that case whenever $x \sim x'$ and $y \sim y'$ we have the following chain of equalities

$$d(x,y) = d(x,y') = d(y',x) = d(y',x') = d(x',y'),$$

showing the independence of d on the choice of representatives x and y.

By triangle inequality we have that

$$d(x,y) \leq d(x,y') + d(y',y) = d(x,y')$$

and
$$d(x,y') \leq d(x,y) + d(y,y') = d(x,y).$$
Thus, $d(x,y) = d(x,y')$, as desired.

(d) All of the axioms of a metric follow by the axioms of a pseudo-metric and a coherent choice of representatives: given $\alpha, \beta, \gamma \in X/\sim$ choose $x \in \alpha$, $y \in \beta$ and $z \in \gamma$. Then

(i) $d^*(\alpha,\alpha) = d(x,x) = 0$. Conversely, if $d^*(\alpha,\beta) = 0$, then $d(x,y) = 0$, so $x \sim y$, so $\alpha = [x] = [y] = \beta$.

(ii) $d^*(\alpha,\beta) = d(x,y) = d(y,x) = d^*(\beta,\gamma)$.

(iii) $d^*(\alpha,\gamma) = d(x,z) \leq d(x,y) + d(y,z) = d^*(\alpha,\beta) + d^*(\beta,\gamma)$.

(e) We show that a map $f : (X^*, d^*) \to \mathbb{R}$ given by

$$f([x]) = x_2$$

is an isometry. We start by noting that f is well-defined, i.e. if x and y are two different representatives of α, then $x \sim y$, so $x_2 = y_2$, hence they given the same $f(\alpha)$ when chosen as representatives for α.

Next, we show that the function f is bijective. The function f is injective because if $f([x]) = f([y])$ then $x_2 = y_2$, so $x \sim y$ and consequently $[x] = [y]$. The function f is surjective because given any $t \in \mathbb{R}$ we have that $f([(0,t)]) = t$.

Finally, the function f preserves distances:

$$|f(x) - f(y)| = |x_2 - y_2| = d^*(x,y).$$

(f) We verify the axioms of a pseudometric for d:

(i) $d(f,f) = |f(x_0) - f(x_0)| + |f(x_1) - f(x_1)| = 0.$

(ii)

$$\begin{aligned} d(f, g) &= |f(x_0) - g(x_0)| + |f(x_1) - g(x_1)| \\ &= |g(x_0) - f(x_0)| + |g(x_1) - f(x_1)| \\ &= d(g, f) \end{aligned}$$

(iii)

$$\begin{aligned} d(f, h) &= |f(x_0) - h(x_0)| + |f(x_1) - h(x_1)| \\ &\leq (|f(x_0) - g(x_0)| + |g(x_0) - h(x_0)|) + (|f(x_1) - g(x_1)| + |g(x_1) - h(x_1)|) \\ &= (|f(x_0) - g(x_0)| + |f(x_1) - g(x_1)|) + (|g(x_0) - h(x_0)| + |g(x_1) - h(x_1)|) \\ &= d(f, g) + d(g, h). \end{aligned}$$

Also, d is not a metric because

$$d(x \mapsto 0, x \mapsto (x - x_0)(x - x_1)) = 0.$$

with $x \mapsto 0$ and $x \mapsto (x - x_0)(x - x_1)$ being two distinct functions $\mathbb{R} \to \mathbb{R}$.

We show that the evaluation function $\mathrm{ev} : \mathcal{F}(\mathbb{R})^* \to (\mathbb{R}^2, d_{\ell^1})$ given by $\mathrm{ev}([f]) := (f(x_0), f(x_1))$ is an isometry. As in the previous part we start by showing that the function ev is well-defined: i.e. if f and g are the representative of the same equivalence class than they give the same element of $\mathbb{R}^2$. If $[f] = [g]$ then $f \sim g$, so

$$0 = d(f, g) = |f(x_0) - g(x_0)| + |f(x_1) - g(x_1)|.$$

Therefore, $(f(x_0), f(x_1)) = (g(x_0), g(x_1))$, so ev is well-defined.

Next we show that the evaluation function ev is bijective. The evaluation function is injective because if $\mathrm{ev}([f]) = \mathrm{ev}([g])$ then $f(x_0) = g(x_0)$ and $f(x_1) = g(x_1)$, so $f \sim g$ and, consequently, $[f] = [g]$. The evaluation function is surjective, because given $(a, b) \in \mathbb{R}$ we have $\mathrm{ev}([f]) = (a, b)$ with

$$f(x) := \begin{cases} a & \text{if } x = x_0, \\ b & \text{if } x = x_1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, ev preserves distances:

$$d_{\ell^1}(\mathrm{ev}([f]), \mathrm{ev}([g])) = |f(x_0) - g(x_0)| + |f(x_1) - g(x_1)| = d^*([f], [g]).$$