Math 171 Homework 4

Due Friday April 29, 2016 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Alex Zamorzaev, in his office, 380-380M (either hand your solutions directly to him, e-mail him, or leave the solutions under his door).

Book problems: Solve Johnsonbaugh and Pfaffenberger, problems 38.4, 39.5, 40.7, 40.14a, 40.17b (note we did c in class), Also solve:

1. **The closure of a set.** Let (M, d) be a metric space and $X \subset M$ a subset. Recall that X is said to be *closed* if $X = \bar{X}$, where $\bar{X}$ is the set of limit points of X in M.

   (a) Prove that for any subset $X \subset M$, $\bar{X}$ is always a closed set. (This justifies our use of the terminology *closure* to refer to $\bar{X}$). (Note: for this problem, you can proceed either directly by definition, or prove that the complement of $\bar{X}$ is always open, making use of Theorem 39.5 from the book).

   (b) Prove that if X is any subset of M with $X \subset Y$ and Y is closed, then $\bar{X} \subset Y$.

   (In some sense, this says that $\bar{X}$ is the *smallest* closed set containing X. Specifically, (b) along with (a) can be used to show that $\bar{X}$ is the intersection of all closed subsets Y containing X).

2. **The interior of a set.** Let (M, d) be a metric space and $E \subset M$ a subset (not necessarily open). An *interior point of* E is a point $p \in E$ such that some $B_\epsilon(p) \subset E$. Define the *interior of* E, denoted $\mathring{E}$, to be the set of all interior points of E.

   (a) Prove that $\mathring{E}$ is open, and that E is open if and only if $\mathring{E} = E$.

   (b) If $G \subset E$ and G is open, prove that $G \subset \mathring{E}$. (In a sense analogous to 1b, this says that $\mathring{E}$ is the largest open set contained in E).

   (c) Prove that the complement of the interior $\mathring{E}^c$ is equal to the closure of the complement $\overline{E^c}$.

3. **The boundary of a set.** If X is a subset of a metric space, define the boundary of X to be the set $\partial X := \bar{X} \cap \overline{X^c}$ (the intersection of the closure of X with the closure of the complement of X). Prove that

   (a) ∂X is closed for any set $X \subset M$.

   (b) $X \cup \partial X = \bar{X}$, for any X.

   (c) $X \backslash \partial X = \mathring{X}$, for any X. (note: ∂X is not necessarily strictly contained in X; here the notation $X \backslash \partial X$ refers to the set of points in X which are not in ∂X).