Math 171 Homework 4

(due April 29)

Problem 38.4.
Let M be a metric space such that M is a finite set. Prove that every subset of M is closed.
Solution:
Every subset of M is finite, so it is closed by Corollary 38.7.
Problem 39.5.
Prove that the interior of a rectangle in $\mathbb{R}^2$

$$(a, b) \times (c, d) = \{(x, y) \mid a < x < b, c < y < d\}$$

is an open subset of $\mathbb{R}^2$.
Solution:
Let $p_0 = (x_0, y_0)$ be a point in $(a, b) \times (c, d)$ and let $\varepsilon = \min(x_0 - a, b - x_0, y_0 - c, d - y_0)$. We will show that $B_\varepsilon(p_0)$ is contained in $(a, b) \times (c, d)$. Indeed, let (x, y) be a point in $B_\varepsilon(p_0)$. Then

$$|x - x_0|^2 \leq |x - x_0|^2 + |y - y_0|^2 = d_{\mathbb{R}^2}((x, y), (x_0, y_0)) \leq \varepsilon^2,$$

and consequently

$$|x - x_0| < \varepsilon.$$

Therefore,

$$a \leq x_0 - \varepsilon < x < x_0 + \varepsilon \leq b.$$

Similarly,

$$|y - y_0| \leq d_{\mathbb{R}^2}((x, y), (x_0, y_0)) < \varepsilon,$$

so that

$$c \leq y_0 - \varepsilon < y < y_0 + \varepsilon \leq d.$$

Thus, $(x, y) \in (a, b) \times (c, d)$, as desired.
Problem 40.7.
Let f be a function from a metric space (M_1, d_1) into a metric space (M_2, d_2). Let $a \in M_1$. Prove that the following are equivalent.

(a) f is continuous at a.

(b) If U is an open subset of M_2 which contains $f(a)$, there exists an open subset V of M_1 which contains a such that $V \subset f^{-1}(U)$.

Solution:

- (a) $\Rightarrow$ (b)

  Assume f is continuous at a. Given an open subset U of M_2 containing $f(a)$, by the openness condition there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(a))$ is contained in U. By continuity of f, there exists $\delta > 0$ such that for all x such that $d_1(a, x) < \delta$ we have that $d_2(f(a), f(x)) < \varepsilon$. Then we can take $V := B_\delta(a)$ because $B_\delta(a)$ is open (Theorem 39.4), $a \in B_\delta(a)$ and

$$f(B_\delta(a)) \subset B_\varepsilon(f(a)) \subset U,$$

so that

$$B_\delta(a) \subset f^{-1}(U).$$

- (b) $\Rightarrow$ (a)

  Assume that for every open subset U of M_2 containing $f(a)$ there exists an open subset V containing a contained in $f^{-1}(U)$.

  Given an arbitrary $\varepsilon > 0$, let $U := B_\varepsilon(f(a))$. By Theorem 39.4 U is open, so there exists an open subset V containing a contained in $f^{-1}(B_\varepsilon(f(a)))$. Since V is open, there exists $\delta > 0$ such that $B_\delta(a) \subset V$. Then

$$B_\delta(a) \subset V \subset f^{-1}(B_\varepsilon(f(a)))$$

  which can be translated into: for all $x \in M_1$ with $d_1(x, a) < \delta$ we have that $d_2(f(x), f(a)) < \varepsilon$. Thus, f is continuous at a, as desired.

Problem 40.14a.

Let (M, d) be a metric space and let X be a subset of M. If $x \in M$, we define

$$d(x, X) := \inf\{d(x, y) \mid y \in X\}.$$

Prove that $f(x) = d(x, X)$ defines a continuous real-valued function on M.

Solution:

It suffices to show that whenever $d(x, x') < \varepsilon/2$ we have that $|d(x, X) - d(x', X)| < \varepsilon$.

Given $x \in M$, choose $y \in X$ such that $d(x, y) < d(x, X) + \varepsilon/2$ (we can do that because by assumption $d(x, X) + \varepsilon/2$ is not an lower bound of the set $\{d(x, y) \mid y \in X\}$). Then for every $x' \in X$ such that $d(x, x') < \varepsilon/2$ we have that

$$d(x', X) \leq d(x', x) + d(x, y) < \frac{\varepsilon}{2} + d(x, X) + \frac{\varepsilon}{2} = d(x, X) + \varepsilon.$$

Similarly, by choosing $y' \in X$ with $d(x', y') < d(x', X) + \varepsilon/2$ we get that

$$d(x, X) < d(x', X) + \varepsilon.$$

Thus,

$$-\varepsilon < d(x, X) - d(x', X) < \varepsilon,$$

as desired.

Problem 40.17b.

Let M be a set and let d and d' be metrics for M. We say that d and d' are *equivalent metrics* for M if the collection of open subsets of (M, d) is identical with the collection of open subsets of (M', d').

Prove that the metrics d, d', and d'' of Exercise 35.7 are equivalent.

Solution:

Lemma 1. *If every sequence $\{a_n\}$ in M that converges in d_1 also converges in d_2 and vice versa then (M, d_1) is equivalent to (M, d_2).*

The desired statement follows from the Lemma 1 applied to the result of Exercise 37.10 from the previous homework.

Proof of Lemma 1. By Theorem 39.5 it suffices to show that a subset X of is closed in (M, d_1) if and only if it is closed in (M, d_2). By symmetry it suffices to only show that if X is closed in (M, d_1), it is closed in d_2.
Assume X is closed in (M, d_1). Let x be a limit point of X in (M, d_2). Then there exists a sequence $\{a_n\}$ in X converging to x in (M, d_2). By the assumption of the problem $\{a_n\}$ also converges to x in (M, d_1), so x is a limit point of X in (M, d_1). Since X is closed in (M, d_1), $x \in X$. Since x was an arbitrary limit point of X in (M, d_2), X is closed in (M, d_2), as desired. □

Problem 1. The closure of a set.
Let (M, d) be a metric space and $X \subset M$ a subset. Recall that X is said to be *closed* if $X = \bar{X}$, where $\bar{X}$ is the set of limit points of X in M.

(a) Prove that for any subset $X \subset M$, $\bar{X}$ is always a closed set. (This justifies our use of the terminology *closure* to refer to $\bar{X}$.)
Note: for this problem you can proceed either directly by definition or prove that the complement of $\bar{X}$ is always open, making use of Theorem 39.5 in the book.

(b) Prove that if X is any subset of M with $X \subset Y$ and Y closed, then $\bar{X} \subset Y$.

Solution:

(a) Let x be a limit point of $\bar{X}$. Thus, there exists a sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ of points in $\bar{X}$ that converges to x. Each $x^{(k)}$ is a limit point of X, so there exists a sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ converging to $x^{(k)}$. Choose N_k such that

$$d(x_n^{(k)}, x^{(k)}) < \frac{1}{k}$$

for all $n \geq N_k$.
Consider the sequence $\{x_{N_k}^{(k)}\}_{k\in\mathbb{N}}$. By the triangle inequality

$$d(x, x_{N_k}^{(k)}) \leq d(x, x^{(k)}) + d(x_n^{(k)}, x^{(k)}).$$

Thus,

$$0 \leq d(x, x_{N_k}^{(k)}) < d(x, x^{(k)}) + \frac{1}{k}. \tag{1}$$

Since, the sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ converges to x, it follows by Theorem 40.3 and Theorem 40.2 that

$$\lim_{k\to\infty} d(x, x^{(k)}) = d(x, x) = 0.$$

Thus, the right-most term of (1) converges to 0. Hence, by Squeeze Theorem

$$\lim_{k\to\infty} d(x, x^{(k)}_{N_k}) = 0.$$

By the definition of a limit, for every $\varepsilon > 0$ there exists N such that for every $k \geq N$ we have that $d(x, x^{(k)}_{N_k}) < \varepsilon$. Thus, the sequence $\{x^{(k)}_{N_k}\}_{k\in\mathbb{N}}$ converges to x, so x is a limit point of X. Thus, $x \in \bar{X}$, as desired.

(b) By definition, $x \in M$ is a limit point of X is there exists a sequence $\{x_n\}$ of points in X converging to x. Since $X \subset Y$, $\{x_n\}$ is also a sequence of points in Y, so x is a limit point of Y. Since Y is closed, $x \in Y$. Thus, Y contains every limit point of X.

Problem 2. The interior of a set.
Let (M, d) be a metric space and $E \subset M$ a subset (not necessarily open). An *interior point of* E is a point $p \in E$ such that some $B_\varepsilon(p) \subset E$. Define the *interior of* E, denoted $\mathring{E}$, to be the set of all interior points of E.

(a) Prove that $\mathring{E}$ is open, and that E is open if and only if $\mathring{E} = E$.

(b) If $G \subset E$ and G is open, prove that $G \subset \mathring{E}$. (In a sense analogous to 1b, this says that $\mathring{E}$ is the largest open set contained in E).

(c) Prove that the complement of the interior $\mathring{E}^c$ is equal to the closure of the complement $\overline{E^c}$.

Solution:

(a) $\mathring{E} = E$
$\Leftrightarrow$ Every point p of E is an interior point.
$\Leftrightarrow$ For every point p of E there exist $\varepsilon > 0$ such that $B_\varepsilon(p) \subset E$.
$\Leftrightarrow$ E is open.

(b) Given any p in G, since G is open there exists $\varepsilon > 0$ such that $B_\varepsilon(p) \subset G$. In particular, $B_\varepsilon(p) \subset E$. Hence, p is an interior point of E. Thus, $G \subset \mathring{E}$.

(c) Since $\mathring{E} \subset E$, we have that $\mathring{E}^c \supset E^c$. Also, because $\mathring{E}$ is open, $\mathring{E}^c$ is closed. Therefore, by Problem 1b, $\overline{E^c} \subset \mathring{E}^c$.

We have that $\overline{E^c} \supset E^c$ which implies $(\overline{E^c})^c \subset E$. Also, because $\overline{E^c}$ is closed, $(\overline{E^c})^c$ is open. Therefore, by part b, $(\overline{E^c})^c \subset \mathring{E}$, which is equivalent to $\overline{E^c} \supset \mathring{E}^c$.

Problem 3. The boundary of a set. If X is a subset of a metric space, define the boundary of X to be the set $\partial X : \bar{X} \cap \overline{X^c}$ (the intersection of the closure of X with the closure of the the complement of X. Prove that

(a) ∂X is closed for any set $X \subset M$.

(b) $X \cup \partial X = \bar{X}$ for any X.

(c) $X \backslash \partial X = \mathring{X}$ for any X.
Note: ∂X is not necessarily strictly contained in X. Here the notation $X \backslash \partial X$ refers to the set of points in X which are not in ∂X.

Solution:

(a) By Problem 1a, both $\bar{X}$ and $\overline{X^c}$ are closed. Hence, ∂X is closed being the intersection of two closed sets.

(b) We have that $X \subset \bar{X}$ and $\partial X \subset \bar{X}$ by definition of ∂X. Hence $X \cup \partial X \subset \bar{X}$. To show that $\bar{X} \subset X \cup \partial X$, it suffices to prove that if x is a limit point of X that is not an element of X then it is an element of ∂X.

Indeed, by assumption, and $x \in X^c$, hence $x \in \overline{X^c}$. Also, by assumption $x \in \bar{X}$. Hence, $x \in \bar{X} \cap \overline{X^c}$, as desired.

(c)
$$X \backslash \partial X = (X^c \cup \partial X)^c$$

By part b applied to X^c (and using that $\partial(X^c) = \partial X$ by definition) the above set equals
$$(\overline{X^c})^c.$$

By Problem 2c, it equals
$$(\mathring{X}^c)^c,$$

which in turn equals $\mathring{X}$, as desired.