Math 171 Homework 5

Due Friday May 6, 2016 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Alex Zamorzaev, in his office, 380-380M (either hand your solutions directly to him or leave the solutions under his door).

Book problems: Solve Johnsonbaugh and Pfaffenberger, problems 34.2, 34.6, 35.9, 41.4, 42.1, 42.2. Also solve:

1. Closures and continuity. Let M and N be metric spaces. Show that the following are equivalent:
   (i) $f : M \to N$ is continuous;
   (ii) $f(\bar{A}) \subset \overline{f(A)}$ for all sets $A \subset M$;
   (iii) $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for all subsets $B \subset N$.

2. Closures and interiors in the relative metric. (note: this is basically book problem 41.5, rewritten with the notation we've been using in class and homework)
   (a) Let M be a metric space and $X \subset M$ a subset endowed with the relative metric. If Y is a subset of X, let $\bar{Y}^X$ denote the closure of Y in the metric space X. Prove that $\bar{Y}^X = \bar{Y} \cap X$.
   (b) Recall that we defined the *interior* B^0 of a set $B \subset N$ in a metric space N on last week's homework. If $Y \subset X \subset M$ as above, state and prove a corresponding result to (a) comparing the interior of Y in X to the interior of Y in M (also, introduce notation for the two different notions.)

3. An interesting example of closed sets in the relative metric. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space with metric $d(p, q) = |p - q|$ (this is the *relative metric* for the inclusion $\mathbb{Q} \subset \mathbb{R}$). Let E be the subset of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$ but that E is not compact. Is E open in $\mathbb{Q}$?

4. (a) Let M be a metric space. A subset $X \subset M$ is said to be *dense* if $\overline{X} = M$. Show that if $X \subset M$ is dense, then for any point $p \in M$ and any $\epsilon > 0$, there exists a point $x \in X$ with $x \in B_\epsilon(p)$.
   (b) A metric space M is said to be *separable* if it contains a countable, dense set. Show that $\mathbb{R}^k$ is separable. *Hint: Let X be the set of points with only rational coordinates.*

5. A collection $\mathcal{V} := \{V_\alpha\}_{\alpha \in I}$ of open subsets of a metric space M is said to be a *base* for M if the following is true: for every $p \in M$ and every open set $U \subset M$ such that $p \in U$, we have $p \in V_\alpha \subset U$ for some α. In other words, every open set in M is the union of a subcollection of the $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint: take all open balls with rational radius whose center lies in some countable dense subset of M.*

6. Prove that every compact metric space K has a countable base, and therefore conclude that K is separable. *Hint: for every $n \in \mathbb{N}$, there are finitely many neighborhoods of radius $1/n$ which cover K.*

7. Constructing open and closed sets. By Theorem 40.5, f is continuous if and only if the preimage under f of any open (respectively closed set) is open (respectively closed). This suggests an easy way to give examples of open and closed sets in a metric space M: write down a function $f : M \to \mathbb{R}$, show that it is continuous, and take the pre-image under f of an open or closed set in $\mathbb{R}$ respectively (then take intersections/unions of such sets ...)... Using this method:

(a) Fix real positive numbers $a_1, \ldots, a_{k+1} > 0$. Show that the generalized ellipsoid $E_{a_1,\ldots,a_{k+1}} := \{(x_1, \ldots, x_{k+1}) | \sum_{i=1}^{k+1} a_i x_i^2 = 1\}$ is a closed subset of $\mathbb{R}^{k+1}$.

(b) Let $V = \{\underline{a} = \{a_n\} \subset \ell^\infty | a_1^2 + a_2 < 1 \text{ and } a_3 > a_4\}$. Show that V is an open subset of ℓ^∞.