# Math 171 Homework 5

(due May 6)

**Problem 34.2.**

(a) Use Exercise 30.8 and the Heine-Borel theorem to prove that if $f$ is continuous on $[a, b]$ and $f(x) > 0$ for every $x$ in $[a, b]$, then there exists $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for every $x \in [a, b]$.

(b) Prove the statement in part (a) by considering the function $g(x) = 1/f(x)$.

**Solution:**

(a) For every $t \in [a, b]$, consider the function $g_t : [a, b] \to \mathbb{R}$ given by $h_t(x) = f(x) - \frac{1}{2}f(t)$. We have that $h_t(t) = \frac{1}{2}f(t) > 0$ is continuous at $x = t$, so by Exercise 30.8 there exists $\delta_t > 0$ such that $h_t(x) > 0$ for every $x \in (t - \delta_t, t + \delta_t) \cap [a, b]$.

We know that the collection open intervals $\{(t-\delta_t, t+\delta_t) \mid t \in [a, b]\}$ covers $[a, b]$, so by Heine-Borel Theorem there exists an finite subcollection $\{(t_i-\delta_{t_i}, t_i+\delta_{t_i}) \mid i = 1, \ldots, n\}$.

Let $\varepsilon = \min\{\frac{1}{2}f(t_i) \mid i = 1, \ldots, n\}$. Then $\varepsilon > 0$ and for every $x \in [a, b]$ there exists an $t_i$ such that $x \in (t_i - \delta_{t_i}, t_i + \delta_{t_i})$, so

$$f(x) - \frac{1}{2}f(t_i) = h_{t_i}(x) > 0$$

and consequently $f(x) > \frac{1}{2}f(t_i) \geq \varepsilon$.

(b) By Theorem 40.4vi, $g$ is continuous on $[a, b]$, so by Theorem 42.6 $g$ is bounded on $M$. Thus, there exists $L > 0$ such that for every $x \in [a, b]$ we have that $g(x) \leq M$. Let $\varepsilon := M$. Then for every $x \in [a, b]$ we have that $f(x) \geq \varepsilon$.

**Problem 34.6. (Uniform continuity).**
Prove that if $f$ is continuous on $[a, b]$ and $\varepsilon > 0$, then there exists $\delta > 0$ such that if $|x-y| < \delta$ and $x, y \in [a, b]$, then $|f(x) - f(y)| < \varepsilon$.
**Solution:**
By definition of continuity we know that for every $x \in [a, b]$ there exists $\delta_x > 0$ such that for every $y \in (x - \delta_x, x + \delta_x) \cap [a, b]$ we have that $|f(y) - f(x)| < \varepsilon/2$. Then for any two element $y$ and $z$ of $(x - \delta_x, x + \delta_x) \cap [a, b]$ we have that

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We know that the collection of open intervals $\{(x-\frac{\delta_x}{2}, x+\frac{\delta_x}{2}) \mid x \in [a, b]\}$ covers $[a, b]$. Hence, by Heine-Borel Theorem, there exists finite subcollection $\{(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \mid i = 1, \ldots, n\}$ that covers $[a, b]$. Let

$$\delta := \min\left\{\frac{\delta_{x_i}}{2} \mid i = 1, \ldots, n\right\}.$$

Then given any $x, y \in [a,b]$ with $|x-y| < \delta$, since the intervals $(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2})$ cover $[a,b]$, there exists $i$ such that $x$ is an element of $(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2})$.
Then

$$|y - x_i| \leq |y - x| + |x - x_i| < \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i},$$

so $y$ lies in $(x_i - \delta_{x_i}, x_i + \delta_{x_i})$. Since $x$ and $y$ are both elements of $(x_i - \delta_{x_i}, x_i + \delta_{x_i})$, we have that $|f(x) - f(y)| < \varepsilon$, as desired.

**Problem 35.9.**

Let $H^\infty$ denote the set of all real sequences $\{a_n\}$ such that $|a_n| \leq 1$ for every positive integer $n$. $H^\infty$ is called the *Hilbert cube.*

(a) Let $\{a_n\}$, $\{b_n\} \in H^\infty$. Prove that the series

$$\sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$$

converges.

(b) Prove that

$$d(\{a_n\}, \{b_n\}) := \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$$

defines a metric on $H^\infty$.

**Solution:**

(a) We have

$$|a_n - b_n| \leq |a_n| + |b_n| \leq 2$$

for every $n$. Hence, the series converges by Comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$.

(b) We verify the axioms of a metric:

- Non-negativity: $d(\{a_n\}, \{b_n\}) \geq 0$, being a sum of a series of non-negative terms.
- Distance to self: $d(\{a_n\}, \{a_n\}) = 0$ because $d(\{a_n\}, \{a_n\})$ is the sum of a series consisting of zeros.
- Non-degeneracy: If $d(\{a_n\}, \{b_n\}) = 0$, then $a_n = b_n$ for every $n$.
- Symmetry: $d(\{a_n\}, \{b_n\}) = d(\{b_n\}, \{a_n\})$ because $|a_n - b_n| = |b_n - a_n|$.
- Triangle inequality: we know that triangle inequality holds term-wise:

$$\frac{|a_n - c_n|}{2^n} \leq \frac{|a_n - b_n|}{2^n} + \frac{|b_n - c_n|}{2^n}$$

for every $n$. Hence, it also holds for partial sums:

$$\sum_{n=1}^{k} \frac{|a_n - c_n|}{2^n} \leq \sum_{n=1}^{k} \frac{|a_n - b_n|}{2^n} + \sum_{n=1}^{k} \frac{|b_n - c_n|}{2^n}.$$

Taking the limit of both sides as $k$ goes to $\infty$ and applying Squeeze Theorem we get the desired triangle inequality for $d$:

$$d(\{a_n\},\{c_n\}) \le d(\{a_n\},\{b_n\}) + d(\{b_n\},\{c_n\}).$$

**Problem 41.4.**
Let $M$ be a metric space and let $X$ be a subset of $M$ with the relative metric. Prove that if $f$ is a continuous function on $M$, then its restriction $f|_X$ to $X$ is a continuous function on $X$.
**Solution:**
We use the sequential characterization of continuity as in Theorem 40.2.
Let $\{x_n\}$ be a sequence of elements of $X$ converging to $x$ in the relative metric. Then $\{x_n\}$ also converges to $x$ viewed as a sequence in $M$, hence the sequence $\{f(x_n)\}$ converges to $f(x)$ which is the same as saying that the sequence $\{f|_X(x_n)\}$ converges to $f|_X(x)$ because $f|_X(y) = f(y)$ for all $y \in X$.
**Problem 42.1.**
Prove that none of the spaces $\mathbb{R}^n, \ell^1, \ell^2, c_0, \ell^\infty$ is compact.
**Solution:**
By Theorem 42.6 it suffices to produce an unbounded continuous function $f$ on each of these spaces to show that they are not compact. We will choose the function of the form $f(x) := d(x,a)$ where $a$ is a fixed point in the respective space. Such an $f$ is continuous by Theorem 40.3.
For $\mathbb{R}^n$ pick $a$ to be the zero vector $\underline{0} = (0,\dots,0)$. For every $L > 0$, let $x := (L+1, 0, 0, \dots, 0)$. Then

$$f(x) = d(x,\underline{0}) = L + 1 > L.$$

Thus, $f$ is unbounded.
For all the other spaces we pick $a$ to be the zero sequence $\underline{0}$: $\underline{0}_n = 0$. For every $L > 0$, let $x$ be the sequence whose first element is $L+1$ and the rest are 0: $x_1 = L+1$ and $x_n = 0$ for $n > 1$. Note that $x$ is an element of each of the spaces $\ell^1, \ell^2$, $c_0$ and $\ell^\infty$.
Then

$$f(x) = d(x,\underline{0}) = L + 1 > L$$

with respect to each of the metrics $\ell^1, \ell^2$ and $\ell^\infty$.
**Problem 42.2.**
Let $X$ be a compact subset of a metric space $M$. Prove that $X$ is closed.
**Solution:**
By Theorem 39.5 it suffices to show that the complement $X^c$ of $X$ is open. Let $y$ be an arbitrary point of $X^c$. Let $f : M \to \mathbb{R}$ be a function given by $f(x) = d(x,y)$. By Theorem 40.3 $f$ is continuous.
Let $U_n := f^{-1}((1/n,\infty))$. By Theorem 40.5iii $U_n$ is open. Also, for every $x \in X$ we have that $d(x,y) > 0$, so there exists $n \in \mathbb{N}$ such that $1/n < d(x,y)$ and consequently $x \in U_n$. Therefore, $\{U_n\}_{n\in\mathbb{N}}$ forms an open cover of $X$. Since $X$ was compact, there exists an open subcover $\{U_{n_k}\}$ of $X$. Let $n_j$ be the largest among $n_k$'s. Then $U_{n_k} \subset U_{n_j}$ for every $k$. Therefore, $X \subset U_{n_j}$, i.e. for all points $x \in X$ we have $d(x,y) > 1/n$.
In particular, for every $z \in M$ such that $d(z,y) < 1/n_j$ we have that $z \in X^c$, so $X^c$ contains an open ball of radius $1/n_j$ around $y$. Thus, $X^c$ is open.

**Problem 1. Closures and continuity.** Let $M$ and $N$ be metric spaces. Show that the following are equivalent:

(i) $f : M \to N$ is continuous;

(ii) $f(\bar{A}) \subset \overline{f(A)}$ for all sets $A \subset M$;

(iii) $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for all subsets $B \subset N$.

**Solution:**

- (ii) $\Rightarrow$ (i) Assume (ii). By Theorem 40.2 to prove continuity of $f$ it suffices to show that the sequence $\{f(x_n)\}$ converges to $f(x)$ for every sequence $\{x_n\}$ of elements of $M$ converging to $x \in M$.

  We will show the previous statement by contradiction. Assume a $\{x_n\}$ in $M$ converges to $x \in M$ and the sequence $\{f(x_n)\}$ *does not* converge to $f(x)$. Then, by definition of convergence, there exists $\varepsilon > 0$ such that for every $N$ there exists $n \geq N$ with $d(f(x_n), f(x)) \geq \varepsilon$.

  Hence, there exists a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ such that $d(f(x_{n_k}), f(x)) \geq \varepsilon$ for every $k$.

  Let $A$ be the set of values of the subsequence $\{x_{n_k}\}$. By construction, for every $p \in A$, $d(f(p), f(x)) \geq \varepsilon$. We know that $\{x_{n_k}\}$ converges to $x$, being a subsequence of $\{x_n\}$. Therefore, $x \in \bar{A}$.

  By assumption (ii), $f(x) \in \overline{f(A)}$. Therefore, there exists a sequence $\{y_n\}$ of elements of $f(A)$ converging to $f(x)$. So, there exists $n$ such that $d(y_n, f(x)) < \varepsilon$. Since $y_n$ is an element of $f(A)$, it can be written as $y_n = f(p)$ for some $p \in A$. However, $d(f(p), f(x)) \geq \varepsilon$, so we got the desired contradiction.

- (iii) $\Rightarrow$ (ii) Assume (iii). Let $A$ be a subset of $M$ and let $x$ be any element of $\bar{A}$. Fix a sequence $\{x_n\}$ of elements of $A$ converging to $x$. It suffices to show that $f(x) \in \overline{f(A)}$.

  Let $B = f(A)$. For each $x_n$, $f(x_n)$ is an element of $f(A)$, so $x_n$ is an element of $f^{-1}(B)$. Therefore, $x \in \overline{f^{-1}(B)}$. By assumption (iii), $x \in f^{-1}(\bar{B})$ which implies $f(x) \in \bar{B}$, as desired.

- (i) $\Rightarrow$ (iii) Assume (i). Let $B$ be a arbitrary subset of $N$ and let $x$ be an arbitrary element of $\overline{f^{-1}(B)}$. Fix a sequence $\{x_n\}$ of element of $f^{-1}(B)$ converging to $x$. By assumption (i), the sequence $\{f(x_n)\}$ converges to $f(x)$. For every $n$, since $x_n \in f^{-1}(B)$ we have that $f(x_n) \in B$. Therefore, the limit $f(x)$ of $\{f(x_n)\}$ is a limit point of $B$: $f(x) \in \bar{B}$. Thus, $x \in f^{-1}(\bar{B})$, as desired.

**Problem 2. Closures and interiors in the relative metric.** (Note: this is basically book problem 41.5 rewritten with the notation we've been using in class and homework.)

(a) Let $M$ be a metric space and $X \subset M$ a subset endowed with the relative metric. If $Y$ is a subset of $X$, let $\bar{Y}^X$ denote the closure of $Y$ in the metric space $X$. Prove that $\bar{Y}^X = \bar{Y} \cap X$.

(b) Recall that we defined the *interior* $\mathring{B}$ of a set $B \subset N$ in a metric space $N$ on last week's homework. If $Y \subset X \subset M$ as above, state and prove a corresponding result to (a) comparing the interior of $Y$ in $X$ to the interior of $Y$ in $M$ (also, introduce notation for the two different notions.)

**Solution:**

(a) We get the desired result via the following sequence of equivalences.

$x \in \bar{Y}^X$.

$\Leftrightarrow$ $x \in X$ and there exists a sequence $\{x_n\}$ in $Y \subset X$ converging to $x$ with respect to the relative metric $d_X$ on $X$.

$\Leftrightarrow$ $x \in X$ and there exists a sequence $\{x_n\}$ in $Y \subset X$ converging to $X$ with to the metric $d_M$ on $M$.

$\Leftrightarrow$ $x \in \bar{Y} \cap X$.

(b) Let $\mathring{Y}^X$ denote the interior of $Y$ in $X$. We show that if $Y \subset X \subset M$ then

$$\mathring{Y} = \mathring{Y}^X \cap \mathring{X}.$$

by proving inclusions both ways.

$\mathring{Y} \subset \mathring{Y}^X \cap \mathring{X}$. Indeed, given $y \in \mathring{Y}$, there exists an open ball $B^M_\varepsilon(y)$ in $M$ that is a subset of $Y$. In particular, $B^M_\varepsilon(y) \subset X$, so $y \in \mathring{X}$. Also, $B^M_\varepsilon(y) \cap X = B^X_\varepsilon(y)$ is the $\varepsilon$-ball in $X$ with the relative metric and it is contained in $Y$. Hence, $y \in \mathring{Y}^X$.

$\mathring{Y} \supset \mathring{Y}^X \cap \mathring{X}$. Given $y \in \mathring{Y}^X \cap \mathring{X}$, we can find an open ball $B^M_{\varepsilon_1}(y)$ which is a subset of $X$. Since $y \in \mathring{Y}^X$, there exists an open ball $B^X_{\varepsilon_2}(y)$ which is a subset of $Y$. Let $\varepsilon = \min\{e_1, e_2\}$. Then any element of $B^M_\varepsilon(y)$ is an element of $X$, hence an element of $B^X_\varepsilon(y)$ and consequently and element of $Y$. Thus, $y \in \mathring{Y}$.

**Problem 3. An interesting example of closed sets in the relative metric.** Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space with the metric $d(p, q) = |p - q|$ (this is the *relative metric* for the inclusion $\mathbb{Q} \subset \mathbb{R}$). Let $E$ be the subset of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that $E$ is closed and bounded in $\mathbb{Q}$ but $E$ is not compact. Is $E$ open in $\mathbb{Q}$?

**Solution:**

Since $\sqrt{2}$ and $\sqrt{3}$ are not in $\mathbb{Q}$, $E$ is the intersection of a closed subset $[-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]$ with $\mathbb{Q}$, and hence closed by Theorem 41.2ii.

It is bounded because for every $p \in E$ we have $|p| < 2$.

To show that $E$ is not compact we construct an infinite open cover that does not have a finite subcover as follows. Let $U_n = (-\infty, \sqrt{3} - 1/n) \cap \mathbb{Q}$. Then each $U_n$ is open in $\mathbb{Q}$ by Theorem 41.2i because it is the intersection of an open set with $\mathbb{Q}$. Together $\{U_n\}$ covers $E$ because every $x \in E$ satisfies $x < \sqrt{3}$, so there exists $n$ such that $1/n < \sqrt{3} - x$ and hence $x \in U_n$. However, no finite subcollection $\{U_{n_k}\}$ covers $E$. Indeed, given any such finite subcollection let $n = \max\{n_k\}$. By denseness of rationals we can find a rational number $q$

in $(\sqrt{3}-1/n, \sqrt{3})$. By construction, this $q$ is an element of $E$ and not an element of any of the $U_{n_k}$. Thus, no finite subcollection $\{U_{n_k}\}$ covers $E$, so $E$ is not compact.
$E$ is a also open in $\mathbb{Q}$ because it is the intersection of an open set $(-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})$ with $\mathbb{Q}$.

**Problem 4.**

(a) Let $M$ be a metric space. A subset $X \subset M$ is said to be *dense* if $\overline{X} = M$. Show that if $X \subset M$ is dense, then for any point $p \in M$ and any $\varepsilon > 0$, there exists a point $x \in X$ with $x \in B_\varepsilon(p)$.

(b) A metric space $M$ is said to be *separable* if it contains a countable, dense set. Show that $\mathbb{R}^k$ is separable. *Hint: Let $X$ be the set of points with only rational coordinates.*

**Solution:**

(a) Let $p$ be an arbitrary point of $M$ and $\varepsilon$ an arbitrary positive number. We have that $p \in \overline{X}$, so there exists a sequence $\{p_n\}$ of elements of $X$ converging to $p$. By the definition of convergence, there exists $N$ such that for every $n \geq N$, $d(x_n, p) < \varepsilon$. In particular, $x_N \in B_\varepsilon(p)$.

(b) We know that $X$ as defined in the hint is countable, being a finite product $\mathbb{Q}^k$ of countable sets $\mathbb{Q}$.

We will show that $X$ is dense in $\mathbb{R}^k$. Let $\underline{x} = (x_1, \ldots, x_k)$ be an arbitrary point of $\mathbb{R}^k$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every coordinate $x_i$ we can pick a sequence $q_i^{(k)}$ in $\mathbb{Q}$ converging to $x_i$. Then by Theorem 37.2 we know that the sequence $\{\underline{q}^{(k)}\}$ converges to $\underline{x}$.

**Problem 5.**

A collection $\mathcal{V} := \{V_\alpha\}_{\alpha \in I}$ of open subsets of a metric space $M$ is said to be a *base* for $M$ If the following is true: for every $p \in M$ and every open set $U \subset M$ containing $p$ there exists $V_\alpha$ containing $p$ and that is a subset of $U$. In other words, every open set in $M$ is the union of a subcollection of the $\{V_\alpha\}_{\alpha \in I}$.
Prove that every separable metric space has a countable base. *Hint: take all open balls with rational radius whose center lies in some countable dense subset of $M$.*

**Solution:**

Let $M$ be a separable metric space. Consider a countable dense subset $X$ of $M$. As suggested in the hint consider the collection $\mathcal{V} := \{B_{1/n}(x) \mid x \in X,\ n \in \mathbb{N}\}\}$. We know that $\mathcal{V}$ is countable because it is equivalent to cartesian product $X \times \mathbb{N}$ of two countable sets. We show that $\mathcal{V}$ forms a base of $M$.
Given $p \in M$ and open set $U$ containing $p$, choose $\varepsilon > 0$ such that $B_\varepsilon(p)$ is contained in $U$. Choose a positive integer $n$ such that $1/n < \varepsilon/2$. By denseness of $X$, we can choose $x \in X$ that is contained in $B_{1/n}(p)$. Then $p \in B_{1/n}(x)$.
If we show that $B_{1/n}(x) \subset B_\varepsilon(p)$ we are done because then $B_{1/n}(x)$ is an element of $\mathcal{V}$ containing $p$ that is a subset of $U$.
Given any $y \in B_{1/n}(x)$, by the triangle inequality

$$d(y,p) \leq d(y,x) + d(x,p) < \frac{1}{n} + \frac{1}{n} < \varepsilon,$$

so $y \in B_\varepsilon(p)$, as desired.

**Problem 6.**

Prove that every compact metric space $K$ has a countable base and therefore conclude that $K$ is separable. *Hint: for every $n \in \mathbb{N}$, there are finitely many neighborhoods of $1/n$ which cover $K$.*

**Solution:**

Fix an arbitrary integer $n$. The collection of open sets $\{B_{1/n}(x) \mid x \in K\}$ covers $K$, so by compactness of $K$ there exists a finite subcollection $\{B_{1/n}(x_k^{(n)}) \mid k = 1, \ldots, m_n\}$ that covers $K$.

Consider the union $\mathcal{V}$ of the finite covers of the form $\{B_{1/n}(x_k^{(n)}) \mid k = 1, \ldots, m_n\}$ over all $n$. We claim that $\mathcal{V}$ forms a base.

Indeed, given any $p \in K$ and an open set $U$ containing $p$, fix an $\varepsilon > 0$ such that $B_\varepsilon(p) \subset U$. Next, fix a positive integer $n$ such that $1/n < \varepsilon/2$. Since $\{B_{1/n}(x_k^{(n)}) \mid k = 1, \ldots, m_n\}$ is a cover of $K$, there exits $k$ such that $B_{1/n}(x_k^{(n)}$ contains $p$.

Then $B_{1/n}(x_k^{(n)} \subset B_\varepsilon(p)$. Indeed, for every $y \in B_{1/n}(x_k^{(n)}$ by the triangle inequality we have that

$$d(y,p) \leq d(y, x_k^{(n)}) + d(x_k^{(n)}, p) < \frac{1}{n} + \frac{1}{n} < \varepsilon.$$

Thus, $\mathcal{V}$ indeed forms a base of $K$. We know that $\mathcal{V}$ is countable, being a countable union of finite sets.

Finally, we show that if $K$ has a countable base $\mathcal{V}$ then $K$ is separable. For every open set $U$ in $\mathcal{V}$ pick an element $x$ of $U$. Let $X$ be the set of the picked elements. Then $X$ is countable because $\mathcal{V}$ is. We show that $X$ is dense, by showing the complement of its closure $(\overline{X})^c$ is empty.

Indeed, assume that $(\overline{X})^c$ is non-empty and pick an element $p \in (\overline{X})^c$. Then there exists an element $U$ of the base containing $p$ that is a subset of $(\overline{X})^c$. However, by construction of $X$, $X$ contains at least one element of $U$ leading to a contradiction.

**Problem 7. Constructing open and closed sets..** By Theorem 40.5, $f$ is continuous if and only if the preimage under $f$ of any open set (respectively, closed set) is open (respectively, closed). This suggests an easy way to give examples of open and closed sets in a metric space $M$: write down a function $f : M \to \mathbb{R}$, show that $f$ is continuous and take the preimage under $f$ of an open or closed set in $\mathbb{R}$. Then we can take intersections/unions of such sets to get even more open and closed sets. Using this method:

(a) Show that the generalized ellipsoid $E_{a_1,\ldots,a_{k+1}} = \{(x_1, \ldots, x_{k+1}) \mid \sum_{i=1}^{k+1} a_i x_i^2 = 1\}$ is a closed subset of $\mathbb{R}^{k+1}$ for $a_1, \ldots, a_{k+1}$ fixed positive real numbers.

(b) Show that $V := \{\underline{a} = \{a_n\} \in \ell^\infty \mid a_1^2 + a_2 < 1 \text{ and } a_3 > a_4\}$ is an open subset of $\ell^\infty$.

**Solution:**

(a) Let $f : \mathbb{R}^{k+1} \to \mathbb{R}$ be given by

$$f(x_1, \ldots, x_{k+1}) = \sum_{i=1}^{k+1} a_i x_i^2.$$

The function $f$ is continuous, being a sum of products of continuous coordinate functions $x_i$. Then $E_{a_1,\ldots,a_{k+1}}$ is closed being the preimage of the closed set $\{1\}$ under $f$.

(b) Let $g, h : \ell^\infty \to \mathbb{R}$ be given by

$$g(\underline{a}) = a_1^2 + a_2$$

and

$$h(\underline{a}) = a_3 - a_4.$$

Again, the functions $g$ and $h$ are continuous because they are polynomials in coordinate functions $a_i$ which are continuous. Then $V$ is the intersection of open sets $g^{-1}((-\infty, 1))$ and $h^{-1}((0, \infty))$ and hence open too.