Math 171 Homework 6

Due Friday May 13, 2016 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Alex Zamorzaev, in his office, 380-380M (either hand your solutions directly to him or leave the solutions under his door).

Book problems: Solve Johnsonbaugh and Pfaffenberger, problems 42.3, 42.8, 42.12, 43.2, 43.7, 44.1, 44.6abcdefg (note for 44.6f you'll need to use the definition of separability given on last week's HW, and the definition of H^∞ given in problem 35.9, which you solved last week).

For Book problem 44.6d, the following definition may be useful.

Definition 1. *A* group *is a set G, equipped with a binary operation $- \cdot - : G \times G \to G$ satisfying the following properties:*

- **closure***: For any two elements c, d in G, $c \cdot d \in G$ as well.*[1]
- **identity element***: There exists an element $e \in G$ with $g \cdot e = e \cdot g = g$ for any element $g \in G$.*
- **associativity***: For any triple of elements $g, h, k \in G$, $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.*
- **existence of inverses***: For any $g \in G$, there exists an element $h \in G$ with $g \cdot h = h \cdot g = e$.*

An example of a group is $\mathbb{R}$ equipped with the operation of addition[2]; this group is further *abelian*, meaning the group operation is commutative: $a \cdot b = b \cdot a$ ($\cdot$ here is used to denote the addition of the real numbers). Note that $\mathbb{R}$ equipped with multiplication is *not a group*, because 0 doesn't have an inverse; however, $\mathbb{R}\backslash\{0\}$ equipped with multiplication is a group. In general, the product structure on a G need not be commutative. For instance, the set of invertible $n \times n$ matrices with real coefficients, called $GL_n(\mathbb{R})$ is another example of a group, where the product of two matrices is their matrix product, which is not often commutative (for instance, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$).

Definition 2. *A* homomorphism of groups $f : G_1 \to G_2$ *is a map of underlying sets $f : G_1 \to G_2$ such that for any $g, h \in G_1$, $f(g \cdot h) = f(g) \cdot f(h)$, where $g \cdot h$ is the multiplication in G_1 and $f(g) \cdot f(h)$ uses the multiplication in G_2.*

An isomorphism of groups $f : G_1 \xrightarrow{\sim} G_2$ *is a homomorphism of groups such that the underlying map of sets is a bijection. Equivalently, G_1 and G_2 are* isomorphic *if there is a pair of homomorphisms $f : G_1 \to G_2$ and $g : G_2 \to G_1$ such that $g \circ f = id_{G_1}$ and $f \circ g = id_{G_2}$.*

Non-book problems:

1. A point x in a metric space is called *isolated* if the set $\{x\}$ is open. Prove that a complete (non-empty) metric space M without isolated points has an uncountable number of

[1] Note: this axiom is implicit in the fact that the group operation is a map $\cdot : G \times G \to G$.

[2] This structure is a part, but not all of the structure $\mathbb{R}$ has as a *field*.

points. **Possible Hint**: Since M is complete, one can produce points in M by producing Cauchy sequences in M. Using this reasoning, it suffices to associate to each element p of an uncountable set P (such as the set of binary sequences $S_{0,1}$), a Cauchy sequence in M such that if $p \neq q$, the Cauchy sequence associated to p must have a different limit from the Cauchy sequence associated to q.

2. If $f : M \to N$ is a function, then recall that *graph of* f is the following subset of $M \times N$

$$\Gamma_f := \{(m, f(m)) | m \in M\}$$

On your midterm exam you were asked to prove that if M and N are metric spaces and f is continuous, then Γ_f is a closed subset of $M \times N$ (equipped with the product metric). This question explores the converse assertion.

(a) It is not always true that Γ_f is closed implies f is continuous. Give an example (with justification) of a non-continuous function f whose graph is closed.

(b) Show that if the target N is a *compact* metric space, and Γ_f is closed, then f is continuous.

3. A metric space M is *locally compact* if every point x has a *compact neighborhood* K, which is by definition a compact set in M whose interior contains x.

(a) Prove that $\mathbb{R}^n$ is locally compact.

(b) Prove $\mathbb{Q}$ is not locally compact, and in fact that no point in $\mathbb{Q}$ has a compact neighborhood.

4. A collection $\mathcal{F}$ of subsets of a set X is said to have the *finite intersection property* if $F_1 \cap \cdots \cap F_n \neq \emptyset$ for any n and any $F_1, \ldots, F_n \in \mathcal{F}$ (i.e., finite intersections in $\mathcal{F}$ are non-empty). Prove that a metric space M is compact if and only if $\cap_{F \in \mathcal{F}} F \neq \emptyset$ for every family $\mathcal{F}$ of closed subsets of M with the finite intersection property.

5. Lebesgue's covering lemma. An important fact that we have proved and used several times in class (without stating the name) is called **Lebesgue's covering Lemma**: If M is a compact metric space and $\mathcal{U}$ is any open cover of M, there is a $\delta > 0$ (depending only on the cover), such that any δ ball $B_\delta(x)$ ($x \in M$) is contained in some element of the cover $\mathcal{U}$. (In your textbook, this appears as Lemma 43.3, as an intermediate step in proving sequential compactness implies compactness. It also is used to show that any continuous function out of a compact metric space is uniformly continuous, see Theorem 44.5). Any such δ which suffices above is called a *Lebesgue number* for the cover $\mathcal{U}$.

(a) Lemma 43.3 in the book uses the sequential compactness property of M to prove Lebesgue's covering Lemma. Give another proof of Lebesgue's covering Lemma directly from the definition of compactness, in terms of every open cover admitting a finite subcover.

(b) Show by example that Lebesgue's covering lemma is false when M is non-compact.