Math 171 Homework 6

(due May 13)

Problem 42.3. Let $X_1, \ldots, X_n$ be a finite collection of compact subsets of a metric space M. Prove that $X_1 \cup X_2 \cup \cdots \cup X_n$ is a compact metric space. Show (by example) that this result does not generalize to infinite unions.

Solution:

Let $\mathcal{U}$ be an open cover of $X_1 \cup \ldots \cup X_n$. Since each X_i is compact, there exists a finite subcollection $\mathcal{U}_i$ of $\mathcal{U}$ that covers X_i. Then $\bigcup_i \mathcal{U}_i$ is a finite subcollection of $\mathcal{U}$ that covers $X_1 \cup \ldots \cup X_n$.

For an counterexample with infinite unions, consider $\bigcup[-1+1/n, 1-1/n] = (0,1)$. Since each interval $[-1+1/n, 1-1/n]$ is closed and bounded, it is compact. However, $(0,1)$ is not closed, so it is not compact.

Problem 42.8. Let f be a continuous, real-valued function on a metric space M which is never zero. Prove that the collection of open sets U for which either $f(x) > 0$ for $x \in U$ or $f(x) < 0$ for $x \in U$ is an open cover of M.

Solution:

It suffices to show that for every $x \in M$ there exists an element U of the collection containing x. Without loss of generality, assume that $f(x) > 0$. Then, by continuity of f there exists a ball $B_\delta(x)$ such that for every $y \in B_\delta(x)$, $f(y) > 0$. Then, $B_\delta(x)$ is the desired element of the collection of open sets.

Problem 42.12. A *contractive mapping* on M is a function f from the metric space (M, d) into itself satisfying

$$d(f(x), f(y)) < d(x, y)$$

whenever $x, y \in M$ with $x \neq y$. Prove that if f is a contractive mapping on a compact metric space M, then there exists a unique point $x \in M$ with $f(x) = x$. (Such a point is called a *fixed point.*)

Solution:

Existence. Consider the function $F : M \to [0, \infty)$ given by

$$F(x) = d(x, f(x)).$$

We show that F is continuous. Fix $x \in M$ and $\varepsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that for every y in $B_\delta(x)$ we have that $d(f(x), f(y)) < \varepsilon/2$. Let $\delta' = \min\{\varepsilon/2, \delta\}$. Then for every y in $B_{\delta'}(x)$ we have that

$$\begin{aligned} d(y, f(y)) &\le d(x, y) + d(x, f(y)) \\ &\le d(x, y) + d(x, f(x)) + d(f(x), f(y)) \\ &< \delta' + d(x, f(x)) + \frac{\varepsilon}{2} \\ &\le \varepsilon + d(x, f(x)). \end{aligned}$$

and by the same argument with x and y switched we also have that

$$d(x, f(x)) < \varepsilon + d(y, f(y)).$$

Thus, for every y in $B_{\delta'}(x)$ we have that

$$|F(x) - F(y)| < \varepsilon,$$

so F is continuous.
Since F is continuous, by Theorem 42.6 $F(x)$ attains a minimum at a point $x_0 \in M$. If $F(x_0) = 0$ then $f(x_0) = x_0$, so we are done. Assume $F(x_0) > 0$. Then, $f(x_0) \neq x_0$, so

$$F(f(x_0)) = d(f(f(x_0)), f(x_0)) < d(f(x_0), x_0) = F(x_0),$$

contradicting the assumption that x was a minimum of F. Thus, $F(x_0) = 0$.
Uniquence. Assume that there are two distinct fixed point x and y of f. Then on one hand

$$d(f(x), f(y)) = d(x, y)$$

because $f(x) = x$ and $f(y) = y$, but on the other hand

$$d(f(x), f(y)) < d(x, y),$$

so we get a contradiction. Thus, the fixed point is unique.
Problem 43.2. Let M_1 and M_2 be compact metric spaces. Prove that the product metric space $M_1 \times M_2$ is compact.
Solution:
By Theorem 43.5, it suffices to show that any sequence $\{(x_n, y_n\}$ in $M_1 \times M_2$ has a convergent subsequence. Since, M_1 is compact, there exists a subsequence $\{(x_{n_k}, y_{n_k}\}$ of $M_1 \times M_2$ whose first coordinates $\{x_{n_k}\}$ converge to some $x \in M_1$. Since, M_2 is compact there exists a further subsequence $\{(x_{n_{k_l}}, y_{n_{k_l}}\}$ of $\{(x_{n_k}, y_{n_k}\}$ whose second coordinates $\{y_{n_{k_l}}\}$ converge to some $y \in M_2$. Since $\{x_{n_{k_l}}\}$ is a subsequence of $\{x_{n_k}\}$, $\{x_{n_{k_l}}\}$ converges to x. Therefore, by Problem 4 on the midterm, $\{(x_{n_{k_l}}, y_{n_{k_l}}\}$ converges to $(x, y) \in M_1 \times M_2$. Thus, the sequence $\{(x_n, y_n\}$ has a convergent subsequence $\{(x_{n_{k_l}}, y_{n_{k_l}}\}$, so $M_1 \times M_2$ is compact.
Problem 43.7. Let X be a compact subset of a metric space M. If $y \in X^c$, prove that there exists a point $a \in X$ such that

$$d(a, y) \leq d(x, y) \qquad \text{for all } x \in X. \tag{1}$$

Give an example to show that the conclusion may fail if "compact" is replaced by "closed".
Solution: Fix $y \in X^c$. Consider the function $f : X \to [0, \infty)$ given by

$$f(x) := d(x, y).$$

We know that f is continuous. Since X is compact, f attains a minimum at some $a \in X$. The point a satisfies (1).
Counterexample for closed. Let $M = (0, 1] \cup (2, 3)$ with the relative metric and let $X = (2, 3)$. Because X is the intersection of a closed subset $[2, 3]$ of $\mathbb{R}$ with M, X is closed in M. Let $y = 1$. For every $a \in X$ we have that $a > 2$. Choose some element $x \in (2, a)$. Then $x \in X$ and

$$d(x, y) = x - 1 < a - 1 = d(a, y).$$

Thus, for X only closed the conclusion of the problem fails.

Problem 44.1. Give an example of metric spaces M_1 and M_2 and a continuous function f from M_1 onto M_2 such that M_2 is compact, but M_1 is not compact.

Solution:

Let $M_1 = \mathbb{R}$, let M_2 consist of a single point p and let $f : M_1 \to M_2$ be given by $f(x) = p$ for all $x \in \mathbb{R}$.

Problem 44.6. Let f be a one-to-one function from a metric space M_1 onto a metric space M_2. If f and f^{-1} are continuous, we say that f is a *homeomorphism* and that M_1 and M_2 are *homeomorphic* metric spaces.

(a) Prove that any two closed intervals of $\mathbb{R}$ are homeomorphic.

(b) Prove (a) with "closed" replaced by "open"; then by "half-open."

(c) Prove that a closed interval is not homeomorphic to either an open interval or a half-open interval.

(d) Let M be a metric space and let $G(M)$ denote the set of homeomorphisms of M onto M.

  (i) Prove that $G(M)$ is a group under composition.

  (ii) Identify the group $G(M)$ in case M is finite.

  (iii) Prove that if M_1 and M_2 are homeomorphic metric spaces, then $G(M_1)$ is isomorphic to $G(M_2)$.

  (iv) Show, by example, that the converse of (iii) does not hold.

(e) Prove that any metric space M is homeomorphic to a metric space (M^*, d) where d is bounded by 1.

(f) Let M be a separable metric space. Prove that there is a one-to-one continuous function f from M to H^∞.

(g) Prove that a metric space M is compact if and only if M is homeomorphic to a closed subset of H^∞.

Solution:

(a) Given two intervals $[a, b]$ and $[c, d]$, the function $f : [a, b] \to [c, d]$ given by

$$f(x) = c + \frac{x-a}{b-a}(d-c) \tag{2}$$

is continuous with a continuous inverse $g : [c, d] \to [a, b]$

$$g(y) = a + \frac{y-c}{d-c}(b-a). \tag{3}$$

(b) The same formulas (2) and (3) define pairs of continuous functions which are inverses of each other: $f_{open} : (a, b) \to (c, d)$ and $g_{open} : (c, d) \to (d, c)$, as well as $f_{left} : [a, b) \to [c, d)$ and $g_{left} : [c, d) \to [a, b)$, as well as $f_{right} : (a, b] \to [c, d)$ and $g_{right} : (c, d] \to [a, b)$.

To show that $[a, b)$ is homeomorphic to $(c, d]$, we consider the continuous function $\tilde{f} : [a, b) \to (c, d]$ given by

$$\tilde{f}(x) = d - \frac{x - a}{b - a}(d - c)$$

and its continuous inverse

$$\tilde{g}(y) = a + \frac{d - y}{d - c}(b - a).$$

(c) Because any closed interval is closed and bounded, it is compact. Because any open interval and any half-open interval is not closed, it is not compact.

(d) (i) We check the axioms of a group for $G(M)$:

* *Closure.* If $f, g : M \to M$ are homeomorphisms then so is $f \circ g$, because $f \circ g$ is continuous and has a continuous inverse $g^{-1} \circ f^{-1}$.
* *Identity element.* The identity function $\mathrm{id}_M : M \to M$ given by $\mathrm{id}_M(x) = x$ satisfies the properties of the identity element of $G(M)$. Namely, for every $f \in G(M)$ and $x \in M$, we have that

$$(f \circ \mathrm{id}_M)(x) = f(\mathrm{id}_M(x)) = f(x)$$

and

$$(\mathrm{id}_M \circ f)(x) = \mathrm{id}_M(f(x)) = f(x)$$

* *Associativity.* For any triple of elements $f, g, h \in G(M)$ and $x \in M$ we have that

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x))) = (f \circ g)(h(x)) = ((f \circ g) \circ h)(x).$$

* *Existence of inverses.* For any $f \in G(M)$, the inverse function $f^{-1} : M \to M$ is continuous and has a continuous inverse f, hence $f^{-1} \in G(M)$. Moreover, f^{-1} satisfies the properties of a group inverse: $f \circ f^{-1} = \mathrm{id}_M$ and $f^{-1} \circ f = \mathrm{id}_M$.

(ii) Let M be a finite set $\{x_1, \ldots, x_n\}$. Any subset of M is open, hence by Theorem 40.5iii, any function $f : M \to M$ is continuous. Therefore, a function $f : M \to M$ is a homeomorphism if and only if it has an inverse function f^{-1}, i.e. if and only if f is a bijection. Such an f is also known as a *permutation*, so $G(M)$ is the symmetric group on an n letters S_n.

(iii) Given a homeomorphism $f : M_1 \to M_2$, one can define an isomorphism $F : G(M_1) \to G(M_2)$ by

$$F(g) = f \circ g \circ f^{-1}$$

for $g \in G(M_1)$. Because f, g, f^{-1} is a homeomorphism, their composition $F(g)$ is also a homeomorphism. The map F has an inverse $F^{-1} : G(M_2) \to G(M_1)$ given by

$$F^{-1}(h) = f^{-1} \circ h \circ f$$

for $h \in G(M_2)$, so F is a bijection between $G(M_1)$ and $G(M_2)$. Finally, F is a group homomorphism because

$$F(g \circ h) = f \circ g \circ h \circ f^{-1} = f \circ g \circ f^{-1} \circ f \circ h \circ f^{-1} = F(g) \circ F(h).$$

(iv) Consider the sets $M_1 = (0,1) \cup \{2\}$ and $M_2 = (0,1)$. Note that M_1 has an isolated point 2 and M_2 has no isolated points, so M_1 and M_2 are not homeomorphic.

We show that the groups $G(M_1)$ and $G(M_2)$ are isomorphic by proving that every element of $G(M_1)$ restricts to an element of $G(M_2)$ and every element of $G(M_2)$ uniquely extends to an element of $G(M_1)$.

Given an element f of $G(M_1)$, we know that $\{f(2)\}$ is the preimage of a an open set $\{2\}$ under the continuous map f^{-1}, so it has to be open in M_1. Thus, $f(2)$ is an isolated point of M_1, so $f(2) = 2$. Therefore the restriction $\tilde{f}$ of f to M_2 defines an a continuous function from M_2 to itself with a continuous inverse $\tilde{f}^{-1}$ being the restriction of f^{-1} to M_2.

Conversely, given any element $g \in G(M_2)$ we can uniquely extend it to $\hat{g} : M_1 \to M_1$ by setting $\hat{g}(2) := 2$. We have that $\hat{g}$ is continuous and has a continuous inverse $\hat{g}^{-1}$ being the extension of g^{-1} to M_1 via $\hat{g}^{-1}(2) = 2$.

(e) Consider the metric d'' from Problem 35.7:

$$d''(x, y) = \min\{d(x, y), 1\}.$$

By construction d'' is bounded by 1. By Problem 37.10, the identity on M viewed as a function from (M, d) to (M, d'') is a homeomorphism.

(f) By part (e) we can assume that d is bounded by 1. Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of M. Define $f : M \to H^\infty$ by

$$f(x) = (d(x, x_1), d(x, x_2), d(x, x_3), \ldots).$$

We prove that f is one-to-one and continuous.

To show that f is one-to-one, it suffices to show that $f(x) \neq f(y)$ whenever $x \neq y$. Assume $x \neq y$ with $x, y \in M$. Then $d(x, y) > 0$. Since $\{x_n : n \in \mathbb{N}\}$ is dense in M, we can choose n such that

$$d(x_n, x) < \frac{d(x, y)}{2}.$$

Since by triangle inequality

$$d(x, y) \leq d(x_n, x) + d(x_n, y),$$

we have that

$$d(x_n, y) \geq d(x, y) - d(x_n, x) > \frac{d(x, y)}{2}.$$

In particular, $d(x_n, x) \neq d(x_n, y)$, so the n^{th} term of $f(x)$ and $f(y)$ is different. Therefore, $f(x) \neq f(y)$.

Next we show that f is continuous. Note that given $\varepsilon > 0$ and $x, y \in M$ such that $d(x, y) < \varepsilon$, by triangle inequality we have that

$$|d(x, x_n) - d(y, x_n)| \leq d(x, y) < \varepsilon$$

for every n. Therefore,

$$d_{H^\infty}(f(x), f(y)) = \sum_{n=1}^{\infty} \frac{|d(x, x_n) - d(y, x_n)|}{2^n} < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus, f is continuous (in fact we just proved that f is uniformly continuous).

(g) Assume that M is compact. Then its image $f(M) \subset H^\infty$ is also compact and hence closed in H^∞. Since f one-to-one, by Theorem 44.3, f is a homeomorphism from M to $f(M)$.

Conversely, to show that every closed subset of H^∞ is compact, by Theorem 43.8 it suffices to show that H^∞ is compact (Problem 43.6).

Let $\{y^{(n)}\}$ be a sequence of elements of H^∞. Let $\mathcal{S}$ be the set of subsequences of $\{y^{(n)}\}$.

Lemma 1. *There exists $x \in H^\infty$, such that for every positive integer k there exists a sequence $\{z^{(n)}\} \in \mathcal{S}$ such that*

$$\lim_{n\to\infty} z_l^{(n)} = x_l$$

for every l from 1 to k.

Assuming the lemma, we construct a subsequence $\{y^{(n_k)}\}$ of $\{y^{(n)}\}$ such that

$$d(y^{(n_k)}, x) < \frac{1}{2^{k-1}}. \tag{4}$$

inductively as follows.

We can choose y^{n_0} to be anything because $d(x, y) \leq 1$ for every pair of points in H^∞. Given $\{y^{(n_l)}\}$ with $l < k$ satisfying the condition (4), by Lemma 1 there exists a subsequence $\{y^{(m_j)}\}$ of $\{y^{(n)}\}$ such that

$$\lim_{j\to\infty} y_l^{(m_j)} = x_l$$

for every l from 1 to k. Then for every l from between 1 and k we can choose N_l such that for every $j \geq N_l$, we have that

$$d(y_l^{(m_j)}, x_l) < \frac{1}{2^k}$$

Let $n_k := \max(\{n_k + 1\} \cup \{N_l \mid l = 1, \ldots, k\})$. Then

$$\begin{aligned} d(y^{(n_k)}, x) &= \sum_{l=1}^{k} \frac{|y_l^{(n_k)} - x_l|}{2^l} + \sum_{l=k+1}^{\infty} \frac{|y_l^{(n_k)} - x_l|}{2^l} \\ &\leq \sum_{l=1}^{k} \frac{1}{2^l} \cdot \frac{1}{2^k} + \sum_{l=k+1}^{\infty} \frac{1}{2^l} \\ &= \frac{1}{2^k}\left(1 - \frac{1}{2^k}\right) + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}, \end{aligned}$$

as desired. The constructed subsequence $\{y^{(n_k)}\}$ of $\{y^{(n)}\}$ converges to x proving the compactness of H^∞.

Proof of Lemma 1. We construct the desired x term by term, using induction on k. For $k = 0$ the statement is vacuous, so we may take $\{z^{(n)}\}$ to be the whole sequence $\{y^{(n)}\}$.

Assume that for an integer $k \geq 1$ we have the first $k-1$ terms $x_1, \ldots, x_{k-1}$ of x and a sequence $\{z^{(n)}\}$ that satisfies the premise of the lemma for $k-1$. Consider the sequence $\{z_k^{(n)}\}$ of k^{th} terms of $\{z^{(n)}\}$. Since $\{z_k^{(n)}\}$ is a sequence on a compact metric space $[0, 1]$, it has a convergent subsequence $\{z_k^{(n_j)}\}$. Let x_k be the limit of $\{z_k^{(n_j)}\}$. Then the sequence $\{z_k^{(n_j)}\} \in \mathcal{S}$ satisfies the premise of the lemma for k. Thus, the inductive step is complete. □

Problem 1. A point x in a metric space is called *isolated* if the set $\{x\}$ is open. Prove that a complete (non-empty) metric space M without isolated points has an uncountable number of points. **Possible Hint:** Since M is complete, one can produce points in M by producing Cauchy sequences in M. Using this reasoning, it suffices to associate to each element p of an uncountable set P (such as the set of binary sequences $S_{0,1}$), a Cauchy sequence in M such that if $p \neq q$, the Cauchy sequence associated to p must have a different limit from the Cauchy sequence associated to q.
Solution:

Lemma 2. *There exists functions from the set of* finite *binary tuples* $f : S_{0,1}^{finite} \to M$ *and* $\varepsilon : S_{0,1}^{finite} \to (0, \infty)$ *such that* $\varepsilon(a) < 1/(length(a) + 1)$ *and for* C_a *being the closed ball of radius* $\varepsilon(a)$ *around* $f(a)$*:*

$$C_a := \{x \in M \mid d(f(a), x) \leq \varepsilon(a)\},$$

the following two statements hold.

- *Whenever a finite binary tuple* a *is a prefix of a finite binary tuple* b *then then* C_a *is contained in* C_b*.*

- *Given two finite binary tuple a and b such that neither is a prefix of the other one, then C_a and C_b are disjoint.*

Assuming Lemma 2, we construct an injective function $F : S_{0,1} \to M$ as follows. Given any element $s \in S_{0,1}$, let its truncation $s^{(k)}$ of length k be the binary k-tuple consisting of the first k elements of s:

$$s^{(k)} = (s_1, \dots, s_k).$$

By Lemma 2, for all $l \geq k$, $f(s^{(l)})$ is contained in a closed ball of radius $\varepsilon(s^{(k)}) < 1/k$ around $s^{(k)}$. Therefore, the sequence $\{s^{(k)}\}_{k\in\mathbb{N}}$ is Cauchy in M and hence converges to some point $F(s)$ in M. Moreover, $F(s)$ is contained in $C_{s(k)}$ for every k.
We show that F is injective. Given two distinct elements s and t of $S_{0,1}$, let k be the first index at such that $s_k \neq t_k$.
Then neither of $s^{(k)}$ and $t^{(k)}$ is a prefix of the other one, hence, by Lemma 2, the balls $C_{s^{(k)}}$ and $C_{t^{(k)}}$ are disjoint. Since, $F(s) \in C_{s^{(k)}}$ and $F(t) \in C_{t^{(k)}}$, it follows that $F(s) \neq F(t)$. Thus, M has an uncountable subset $F(S_{0,1})$ and, hence, is uncountable.

Proof of Lemma 2. We construct f and ε by induction on the length l of the finite binary tuples.
For $l = 0$ we have a single 0-tuple – the empty tuple () to which we associate some element $x \in M$: $f(()) = x$. Let $\varepsilon(()) = 1/2$.
Assume that we have constructed f and ε for all binary l tuples with $l \leq k$. Fix a k-tuple $a = (a_1, \dots, a_k)$. Next we define f on $a0 = (a_1, \dots, a_k, 0)$ and $a1 = (a_1, \dots, a_k, 1)$.
Let $f(a0) = f(a)$. Since $f(a)$ is not an isolated point, there exists a point $y \in M$ such that $d(f(a), y) < \varepsilon(a)/2$. Let $f(a1) = y$ and let

$$\varepsilon(a0) = \varepsilon(a1) = \min\left\{\frac{d(f(a), y)}{3}, \frac{1}{k+3}\right\}.$$

By construction C_{a0} and C_{a1} a disjoint and both contained in C_a and $\varepsilon(a0)$ and $\varepsilon(a1)$ are both smaller than $1/(k+2)$.
This way we define f and ε on all $(k+1)$-tuples. It remains to check that f and ε still satisfy the two conditions of the lemma.
Given a prefix $a^{(l)} = (a_1, \dots, a_l)$ of $a^{(k+1)}(a_1, \dots, a_{k+1})$, by inductive assumption $C_{a^{(k)}}$ is contained $C_{a^{(l)}}$. By the discussion in the previous paragraph $C_{a^{(k+1)}}$ is contained in $C_{a^{(k)}}$, so $C_{a^{(k)}}$ is contained in $C_{a^{(l)}}$.
Given two tuples a and b of length at most $k + 1$ that are not prefixes of one another, we have three cases:

- Case 1: the largest common prefix of a and b is of length $l < k$. Then the tuples $a^{(l+1)}$ and $b^{(l+1)}$ of the first $(l + 1)$ elements of a and b are not prefixes of one another, so by inductive assumption $C_{a^{(l+1)}}$ and $C_{b^{(l+1)}}$ are disjoint. Since C_a and C_b are contained in $C_{a^{(l+1)}}$ and $C_{b^{(l+1)}}$, respectively, they are also disjoint.

- Case 2: the largest common prefix c of a and b is of length k. Then then $\{a, b\} = \{c0, c1\}$ and we already mentioned that C_a and C_b are disjoint right after the construction.

□

Problem 2. If $f : M \to N$ is a function, then recall that the *graph of* f is the following subset of $M \times N$:

$$\Gamma_f := \{(m, f(m)) \mid m \in M\}.$$

On your midterm exam you were asked to prove that if M and N are metric spaces and f is continuous, then Γ_f is a closed subset of $M \times N$ (equipped with the product metric). This question explores the converse assertion.

(a) It is not always true that Γ_f is closed implies f is continuous. Give an example (with justification) of a non-continuous function f whose graph is closed.

(b) Show that if the target N is a *compact* metric space, and Γ_f is closed, then f is continuous.

Solution:

(a) Consider $f : [0, \infty) \to [0, \infty)$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} & \text{if } x > 0. \end{cases}$$

We know that g is not continuous, because $\lim_{x \to 0} f(x)$ does not exist.

The graph of f is s a union of $\{(0, 0)\}$ and the graph of $g : (0, \infty) \to [0, \infty)$ given by $g(x) = 1/x$. Hence, if Γ_g is closed in $[0, \infty) \times [0, \infty)$, then so is Γ_f.

Let (x, y) be a limit point of Γ_g in $[0, \infty)$. Then there exists a sequence $\{(x_i, y_i)\}$ in Γ_g converging to (x, y). Then $\{x_n\}$ and $\{y_n\}$ converge to x and y, respectively, in $[0, \infty)$. Hence, on one hand, $x_i y_i$ converges to xy. On the other hand, since $\{(x_i, y_i)\} \in \Gamma_g$, we have that $x_i y_i = 1$, so $\{x_n y_n\}$ converges to 1. Thus, $xy = 1$, so $x > 0$ and $y = 1/x$, i.e. $(x, y) \in \Gamma_g$. Thus, Γ_g contains all of its limit points in $[0, \infty) \times [0, \infty)$.

(b) Assume N is compact, Γ_f is closed f is not continuous to get a contradiction. Then there exists a sequence $\{x_n\}$ in M converging to some $x \in M$ such that $\{f(x_n)\}$ does not converge to $f(x)$. Therefore, there exists $\epsilon > 0$ such that for infinitely many n,

$$d(f(x_n), f(x)) \geq \epsilon.$$

Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(f(x_{n_k}), f(x))) \geq \epsilon, \ \forall k. \tag{5}$$

Since N is compact, there exists a subsequence $\{f(x_{n_{k_l}})\}$ of $\{f(x_{n_k})\}$ that converges to some $n \in N$. Then the sequence of points $\{(x_{n_{k_l}}, f(x_{n_{k_l}})\}$ converges to $(x, n) \in M \times N$. Since each $\{(x_{n_{k_l}}, f(x_{n_{k_l}})\}$ is a point on the graph Γ_f and Γ_f is closed, $(x, n) \in \Gamma_f$, i. e. $n = f(x)$. However, by assumption (5) $\{f(x_{n_{k_l}})\}$ cannot converge to $f(x)$ leading to a contradiction.

Problem 3.
A metric space M is *locally compact* if every point x has a *compact neighborhood* K which is by definition a compact set in M whose interior contains x.

(a) Prove that $\mathbb{R}^n$ is locally compact.

(b) Prove that $\mathbb{Q}$ is not locally compact, and in fact that no point in $\mathbb{Q}$ has a compact neighborhood.

Solution:

(a) Every point $x \in \mathbb{R}^n$ is contained in a compact set $[x-1, x+1]$ whose interior $(x-1, x+1)$ contains x.

(b) Assume that $x \in \mathbb{Q}$ has a compact neighborhood K. Since the interior of K contains x, $B_\varepsilon^{\mathbb{Q}}(x) \subset K$ for some $\varepsilon > 0$. By denseness of irrational numbers, we can pick an irrational number $y \in (x - \varepsilon, x + \varepsilon)$.

By denseness of rationals we can pick a sequence of rationals $\{q_n\}$ in $(x - \varepsilon, x + \varepsilon)$ converging to y. Then, by compactness of K, $\{q_n\}$ has a subsequence $\{q_{n_k}\}$ converging to some $z \in K$ in the relative metric. Since the relative metric is the restriction of the Euclidean metric, $\{q_{n_k}\}$ converges to z in $\mathbb{R}$. However, since we assumed that $\{q_n\}$ converges to y in $\mathbb{R}$, $z = y$ which contradicts our assumptions that y is irrational and z is rational. Thus, x has not compact neighborhood.

Problem 4. A collection $\mathcal{F}$ of subsets of a set X is said to have the *finite intersection property* if $F_1 \cap \cdots \cap F_n \neq \emptyset$ for any n and any $F_1, \ldots, F_n \in \mathcal{F}$ (i.e. finite intersections in $\mathcal{F}$ are non-empty). Prove that a metric space M is compact if and only if $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$ for every family $\mathcal{F}$ of closed subsets of M with the finite intersection property.
Solution:
We have the following chain of equivalences.

M is compact.

$\Leftrightarrow$ Every family $\mathcal{U}$ of open sets in M such that $\bigcup_{U \in \mathcal{U}} U = M$ has a finite subfamily $U_1, \ldots, U_n$ such that $\bigcup_{i=1}^n U_i = M$.

$\Leftrightarrow$ If $\mathcal{U}$ is a family of open sets in M such that every finite subfamily $U_1, \ldots, U_n$ of $\mathcal{U}$ satisfies $\bigcup_{i=1}^n U_i \neq M$ then $\bigcup_{U \in \mathcal{U}}^n U \neq M$.

$\Leftrightarrow$ If $\mathcal{U}$ is a family of open sets in M such that every finite subfamily $U_1, \ldots, U_n$ of $\mathcal{U}$ satisfies $\bigcap_{i=1}^n U_i^c \neq \emptyset$ then $\bigcap_{U \in \mathcal{U}}^n U^c \neq \emptyset$.

$\Leftrightarrow$ If $\mathcal{F}$ is a family of closed sets in M such that every finite subfamily $F_1, \ldots, F_n$ of $\mathcal{F}$ satisfies $\bigcap_{i=1}^n F_i \neq \emptyset$ then $\bigcap_{F \in \mathcal{F}}^n F \neq \emptyset$.

Problem 5. Lebesgue's Covering Lemma. An important fact that we have proved and used several times in class (without stating the name) is call **Lebesgue's Covering Lemma**: If M is a compact metric space and $\mathcal{U}$ is any open cover of M, then there exists a $\delta > 0$ (depending only on the cover), such that any δ-ball $B_\delta(x)$ (with $x \in M$) is contained in some element U of $\mathcal{U}$. (In your textbook, this appears as Lemma 43.3, as an intermediate step in proving sequential compactness implies compactness. It also is used to show that any continuous function from a compact metric space is uniformly continous, see Theorem 44.5). Any such δ which satisfies the condition above is called a *Lebesgue number* for the cover $\mathcal{U}$.

(a) Lemma 43.3 in the book uses the sequential compactness property of M to prove Lebesgue's covering lemma. Give another proof of Lebesgue's covering lemma directly from the definition of compactness, in terms of every open cover admitting a finite subcover.

(b) Show by example that Lebesgue's covering lemma is false when M is non-compact.

Solution:

(a) Because $\mathcal{U}$ is a cover of M, for every $x \in M$ we can choose an element U_x of $\mathcal{U}$ that contains x. Because U_x is open, we can choose $\delta_x > 0$ such that $B_{\delta_x}(x)$ is contained in U_x. The collection

$$\{B_{\delta_x/2}(x) \mid x \in M\}$$

forms an open cover of M. By compactness of M, there exists a finite subcollection $\{B_{\delta_{x_i}/2}(x_i) \mid i = 1, \ldots, n\}$ that covers n. We claim that

$$\delta := \min\{\frac{\delta_{x_i}}{2} \mid i = 1, \ldots, n\}.$$

is a Lebesgue number of $\mathcal{U}$. Indeed, given any $x \in M$, by assumptions, there exists i such that $x \in B_{\delta_{x_i}/2}(x_i)$ and a $U \in \mathcal{U}$ such that $B_{\delta_{x_i}}(x_i) \subset U$.

Then for any $y \in B_\delta(x)$ we have that

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \delta + \frac{\delta_i}{2} \leq \delta_i.$$

Hence, $B_\delta(x) \subset B_{\delta_i}(x_i) \subset U$, as desired.

(b) For example, take $M := (0, \infty)$ and

$$\mathcal{U} := \{(r, 3r) \mid r \in (0, \infty)\}.$$

The collection $\mathcal{U}$ forms an open cover of M because for every $x \in M$, $x \in (r, 2r)$ for $r = 2x/3$.

However, we show that for every $\delta > 0$, there exists $x \in M$ such that the ball $B_\delta(x)$ is not contained in any of the elements of M. Given $\delta > 0$, let $x = \delta/3$. Then if an element $(r, 2r)$ of $\mathcal{U}$ contains x, then $r < \delta/3$, so $2r < 2\delta/3$. In particular, the element $x + 2\delta/3 = \delta$ of $B_\delta(x)$ is not contained in $(r, 2r)$.