Math 171 Homework 7

(due May 20)

Problem 45.2.

(a) Give an example or a subset of $\mathbb{R}$ which is connected but not compact.

(b) Give an example of a subset of $\mathbb{R}$ which is compact but not connected.

(c) Characterize the compact connected subsets of $\mathbb{R}$.

Solution:

(a) $\mathbb{R}$ is connected by Theorem 45.7 and not compact by Theorem 43.9.

(b) $\{0, 1\}$ is not connected because the point $\{0\}$ is open and closed in $\{0, 1\}$ and compact because it is finite.

(c)

Claim 1. *The compact connected subsets of* $\mathbb{R}$ *are*

- *the empty set,*
- *singleton sets* $\{x\}$, $x \in \mathbb{R}$ *and*
- *closed intervals* $[a, b]$, $a, b \in \mathbb{R}$, $a < b$.

Proof. By Corollary 45.4, the connected subsets of $\mathbb{R}$ are the empty set, the singleton sets, bounded intervals (open, closed and half-open), rays ($[a, \infty)$ and $(-\infty, a]$) and $\mathbb{R}$ itself. By Theorem 43.9 the compact subsets of $\mathbb{R}$ are the closed bounded sets. Open intervals are not closed. Rays and $\mathbb{R}$ are not bounded. Hence, we get the list in the claim. □

Problem 45.5. Let X be a connected subset of a metric space M. Prove that $\overline{X}$ is connected. Is $\mathring{X}$ necessarily connected?

Solution: Assume that $\overline{X}$ is not connected. We will show that X is also not connected. Write $\overline{X} = U \cup V$ where U and V are disjoint non-empty open subsets of $\overline{X}$. Then $X = (X \cap U) \cup (X \cap V)$ with $X \cap U$ and $X \cap V$ disjoint open in X. To show that X is not connected it suffices to show that $X \cap U$ and $X \cap V$ are non-empty. Assume that $X \cap U$ is empty. Then $X = X \cap V$, so $X \subset V$. We show that this implies that $\bar{X} = V$ contradicting our assumptions that U in nonempty. Indeed, let x be a limit point of X and let $\{x_n\}$ be a sequence of elements of X converging to x. Then each x_n is contained in V. Since V is closed in $\bar{X}$, and x is a limit point of V, $x \in V$.

Next we give an example when X is connected and $\mathring{X}$ is not connected. Let

$$X = ([-2, 0] \times [-2, 0]) \cup ([0, 2] \times [0, 2]).$$

We show that $\mathring{X}$ is not connected. Because for every $\varepsilon > 0$, the point $(-\varepsilon/2, \varepsilon/2)$ is an element of $B_\varepsilon((0, 0))$ and not of X, $(0, 0) \notin \mathring{X}$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x + y$. Since $(0, 0) \notin \mathring{X}$, $f(x, y) \neq 0$ for every $(x, y) \in \mathring{X}$.
Since f is continuous, the set $U := f^{-1}((0, \infty))$ is open in $\mathbb{R}^2$. Therefore, $U \cap \mathring{X}$ is open in $\mathring{X}$. The set $U \cap \mathring{X}$ is nonempty because it contains the point $(1, 1)$. The set $U \cap \mathring{X}$ is not all of $\mathring{X}$ because it does not contain the point $(-1, -1) \in \mathring{X}$. Finally, $U \cap \mathring{X}$ is closed in $\mathring{X}$ because $U \cap \mathring{X} = f^{-1}([0, \infty)) \cap \mathring{X}$. Thus, $\mathring{X}$ is not connected.
Problem 45.7.

(a) Show, by example, that unions and intersections of connected sets are not necessarily connected.

(b) Prove that if X and Y are connected subsets of $\mathbb{R}$, then $X \cap Y$ is connected.

Solution:

(a) Consider connected singleton sets $\{0\}$ and $\{1\}$. Their union $\{0, 1\}$ is not connected.

(b) By Theorem 45.3, it suffices to show that whenever $a, b \in X \cap Y$ with $a < b$ then $[a, b] \subset X$. Assume that $a, b \in X \cap Y$. Since $a, b \in X$ and X is connected, by Theorem 45.3, $[a, b] \subset X$. Similarly, $[a, b] \subset Y$. Thus, $[a, b] \subset X \cap Y$, as desired.

Problem 46.2. Give an example of a complete metric space that is not compact.
Solution:
$\mathbb{R}$.
Problem 46.3. Given an example of a connected metric space that is not complete.
Solution: $(0, 1)$ is connected by Corollary 45.4, but not complete because the Cauchy sequence $\{1/n\}_{n>1}$ does not converge in $(0, 1)$.
Problem 46.5. Let M be a metric space.

(a) Prove that if C is a complete subset of M, then C is closed.

(b) Prove that if M is complete, then every closed subset of M is complete.

Solution:

(a) Let x be limit point of C in M. Let $\{x_n\}$ be a sequence of elements of C converging to an element x of M. By Theorem 46.2, $\{x_n\}$ is a Cauchy sequence in M. Since each x_n is an element of C, $\{x_n\}$ is a Cauchy sequence in C. Since C is complete, $\{x_n\}$ converges to some $y \in C$ as a sequence in C. Thus, $\{x_n\}$ converges to both x and y as a sequence in M, so $x = y \in C$. Thus, C is contains all of its limit points.

(b) Assume M is complete and let C be a closed subset of M. Let $\{x_n\}$ be a Cauchy sequence in C. Since M is complete, $\{x_n\}$ converges to some $x \in M$. Since C is closed $x \in C$. Thus, every Cauchy sequence in C converges to an element of C.

Problem 1. Let X be any two element set, for instance $\{1, 2\}$, endowed with the discrete metric. Prove that a metric space M is connected if and only if continuous functions $f : M \to X$ are constant.

Solution:

Assume that there exists an non-constant function $f : M \to X$. Since the singleton sets $\{1\}$ and $\{2\}$ are open in the discrete metric and f is continuous it follows that $f^{-1}(\{1\})$ and $f^{-1}(\{2\})$ are open subsets of M. Since $X = \{1\} \cup \{2\}$ it follows that

$$M = f^{-1}(\{1\}) \cup f^{-1}(\{2\}).$$

Since f is non-constant, both $f^{-1}(\{1\})$ and $f^{-1}(\{2\})$ and are non-empty. Thus, M can be written as a union of two disjoint nonempty open sets $f^{-1}(\{1\})$ and $f^{-1}(\{2\})$, so M is not connected.

Assume M is not connected. Say $M = U \cup V$ with U and V disjoint nonempty open subsets of M. Define $f : M \to X$ by

$$f(x) \begin{cases} 1 & \text{if } x \in U, \\ 2 & \text{if } x \in V. \end{cases}$$

Then f is continuous because the preimage of every open subset of X is open in M. Indeed, X has 4 subsets: $\emptyset$, $\{1\}$, $\{2\}$, X, each of them open in X. The preimage of each of the 4 subsets of X is open in M: $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{1\}) = U$, $f^{-1}(\{2\}) = V$, $f^{-1}(X) = M$.