Math 171 Homework 8

Due Friday May 27, 2016 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Alex Zamorzaev, in his office, 380-380M (either hand your solutions directly to him or leave the solutions under his door).

For problems 1 and 2 below, you may use the following facts about *measure zero sets* without justification:

Fact 1 **If $A \subset \mathbb{R}^n$ is a set of measure zero, and $B \subset A$ is any subset, then B has measure zero too.** The proof of this is very straightforward, following from the fact that if $A \subset \cup_{i \in \mathbb{N}} I_i$, then $B \subset A \subset \cup_{i \in \mathbb{N}} I_i$.

Fact 2 **If U is any non-empty open set in $\mathbb{R}^n$ (or in an interval R with non-empty interior), then U does *not* have measure zero.** *Some remarks about the proof*: by applying Fact 1, since any open set contains a small open interval, it suffices to show that a small non-empty open interval in $\mathbb{R}^n$ is not of measure zero. In turn, any small open interval contains a smaller closed interval with non-zero volume, so it suffices to show that a closed interval I with volume $V := |I| > 0$ has non-zero measure. On problem 6 below, you will prove such a statement for a closed interval in $\mathbb{R}$, and the proof for a closed interval in $\mathbb{R}^n$ is similar, though somewhat more tedious.[1]

A corollary of Fact 1 and Fact 2 is this: *If A is a measure zero subset $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ is a non-empty open set, then $U \cap A \neq U$, meaning $U \backslash (U \cap A)$ is non-empty.*

Non-book problems:

1. Let R be a closed and bounded interval in $\mathbb{R}^n$, and let $f : R \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ at almost every $x \in R$. Prove that $f(x) = 0$ for all $x \in R$.

Note/Hint: Making use of the above Facts 1 and 2 will likely be very helpful.

2. Let R be a closed and bounded interval in $\mathbb{R}^n$, and let $f, g : R \to \mathbb{R}$ be two Riemann integrable functions on R. Suppose that $f(x) = g(x)$ almost everywhere in $x \in R$. Prove that $\int_R f = \int_R g$.

Detailed hint: Here is a sketch of one approach to prove this:

(a) First, since f and g are Riemann integrable, so is $h := f - g$; (why?) Hence, show that it suffices to establish that if h is a Riemann integrable function on R which is zero almost everywhere, then $\int_R h = 0$.

(b) Suppose now that h is Riemann integrable and equal to zero almost everywhere. Then, show that the lower integral $\underline{\int}_R h \leq 0$. Similarly, show that the upper integral

[1] A sharper statement is that if I is closed interval with volume V, then it has *outer measure* V, meaning that $\inf\{\sum_{i \in \mathbb{N}} |J_i| | \{J_i\}$ a countable collection of open intervals whose union contains $I\} = V$. The notion of *outer measure* is often developed in Math 172, as part of a systematic study of measure theory and Lebesgue integration.

$\overline{\int}_R h \geq 0$; since h is integrable it would follow that $\int_R h = 0$. The following observation is crucial to showing, for instance that $\underline{\int}_R h \leq 0$ (and may require Facts 1 or 2 above): If φ is a step function on R adapted to some partition $\mathcal{P}$ with $\varphi \leq h$ everywhere, then for every $I \in \mathcal{P}$, $\varphi|_{\mathring{I}} \leq 0$ and hence $\int_R \varphi \leq 0$ (why?). Conclude the argument.

3. (i) Let R be a closed and bounded interval in $\mathbb{R}^n$, and let φ and ψ be two step functions on R; that is, $\varphi, \psi \in \mathcal{S}(R)$. Prove a statement we asserted in class, that $\min(\varphi, \psi)$ is again a step function on R.

(ii) Deduce another statement we asserted in class: that if $f, g \in \mathcal{L}_+(R)$, then $\min(f, g) \in \mathcal{L}_+(R)$.

4. Let f be an *increasing* function on the closed interval $[a, b] \in \mathbb{R}$. Prove that f is Riemann integrable.

5. Define the *Cantor set* $C \subset [0, 1]$ to be the set of real numbers in $[0, 1]$ whose base-3 expansions do not contain a 1. That is,

$$C := \{x \in [0, 1] | x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \text{with each } a_i \in \{0, 2\}\}.$$

(i) Show that C is uncountable.

(ii) Show that C has Lebesgue measure zero. **Hint**: One can write $C := \cap_n C_n$, where C_n is the set of real numbers $x \in [0, 1]$ such that the first n digits of base-3 expansion of x does not contain a 1. This leads to another way of thinking about the Cantor set, as being formed from $[0, 1]$ by successively removing intervals. . .

6. Show that if $\{I_j\}_{j \in \mathbb{N}}$ is a collection of open intervals in $\mathbb{R}$ which covers $[0, 1]$, meaning that $[0, 1] \in \cup_{j=1}^{\infty} I_j$ then $\sum_{j=1}^{\infty} |I_j| \geq 1$. Deduce that $[0, 1]$ does *not* have Lebesgue measure zero. **Hint:** use compactness.