Math 171 Homework 8

(due May 27)

Problem 1. Let R be a closed and bounded interval in $\mathbb{R}^n$, and let $f : R \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ at almost every $x \in R$. Prove that $f(x) = 0$ for all $x \in R$.

Solution:

Let

$$A := \{x \in \mathbb{R} \mid f(x) \neq 0\}.$$

By assumption A has measure 0. Since f is continuous and A is the preimage of the open set $\mathbb{R}\{0\}$, A is open. Since A is an open subset of $\mathbb{R}^n$ that has measure zero, by Fact 2 A is empty. Thus, $f(x) = 0$ for all $x \in R$.

Problem 2. Let R be a closed and bounded interval in $\mathbb{R}^n$ and let $f, g : R \to \mathbb{R}$ be two Riemann integrable functions on R. Suppose that $f(x) = g(x)$ almost everywhere in $x \in R$. Prove that $\int_R f = \int_R g$.

Solution:

Consider $h := f - g$. By assumption $h(x) = 0$ almost everywhere on R. Since the Riemann integrable functions form a vector space, h is Riemann integrable.

Let φ be an a step function adapted to some partition $\mathcal{P}$ such that $\varphi \leq h$. Let A be the set of $x \in R$ such that $h(x) \neq 0$. By assumption A is measure zero. For every interval I of the partition $\mathcal{P}$, by Fact 2, $\mathring{I}$ is not measure zero. Hence, by Fact 1, $\mathring{I}$ is not a subset of A. Therefore, there exists $x \in \mathring{I}$ such that $h(x) = 0$. For such an x we have that $\varphi(x) \leq 0$. Since φ is constant on $\mathring{I}$, $\varphi(x) = a_I \leq 0$ for all $x \in \mathring{I}$. Since I was an arbitrary element of R we have $\varphi \leq 0$ on the interiors of all intervals in R (i.e. $a_I \leq 0$ for every $I \in R$). Therefore,

$$\int_R \varphi = \sum_{I \in R} a_I \cdot \text{volume}(I) \leq 0.$$

Since $\int_R \varphi \leq 0$ for every step function on R such that $\varphi \leq h$, we have that

$$\underline{\int}_R h \leq 0.$$

Applying the same argument to $-h$ we get that

$$\overline{\int}_R h \geq 0.$$

Since h is Riemann integrable,

$$\underline{\int}_R h = \overline{\int}_R h.$$

Thus,

$$\int_R h = 0.$$

By Theorem 2.3 in Leon Simon's notes, we have that

$$\int_R h = \int_R f - \int_R g,$$

so $\int_R f = \int_R g$.

Problem 3.

(i) Let R be a closed and bounded interval in $\mathbb{R}^n$, and let φ and ψ be two step functions on R; that is, $\varphi, \psi \in \mathcal{S}(R)$. Prove a statement we asserted in class, that $\min(\varphi, \psi)$ is a again a step function on R.

(ii) Deduce another statement we asserted in class: that if $f, g \in \mathcal{L}_+(\mathbb{R})$, then $\min(f, g) \in \mathcal{L}_+(R)$.

Solution:

(i) Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions associated with φ and ψ and let $\mathcal{R}$ be the common refinement of $\mathcal{P}$ and $\mathcal{Q}$. Then for every interval $I \in \mathcal{R}$, both φ and ψ are constant on $\mathring{I}$, so $\min(\varphi, \psi)$ is also constant on $\mathring{I}$. Thus, $\min(\varphi, \psi)$ is a step function with partition $\mathcal{R}$.

(ii) Since f and g are elements of $\mathcal{L}_+(\mathbb{R})$, it follows that there exist increasing sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of non-negative step functions such that $f(x) = \lim_{n\to\infty} \varphi_n(x)$ on $R \backslash A$ and $g(x) = \lim_{n\to\infty} \psi_n(x)$ on $R\backslash B$ (where A and B are sets of measure zero).

We show that $\min(f, g)(x) = \lim_{n\to\infty} \min(\varphi_n, \psi_n)(x)$ on $R\backslash(A\cup B)$. Fix $x \in R\backslash(A\cup B)$. Consider two cases:

- Case 1: $f(x) = g(x)$. Then $\min(f, g)(x) = f(x) = g(x)$. Given an $\varepsilon > 0$, choose N_1 and N_2 such that $|\varphi_n(x) - f(x)| < \varepsilon$ for every $n \geq N_1$ and $|\psi_n(x) - g(x)| < \varepsilon$ for every $n \geq N_2$. Then for every $n \geq \max(N_1, N_2)$ we have that

$$|\min(\varphi_n(x), \psi_n(x)) - f(x)| < \varepsilon,$$

so $\lim_{n\to\infty} \min(\varphi_n, \psi_n)(x) = f(x)$, as desired.

- Case 2: $f(x) \neq g(x)$. Without loss of generality assume that $f(x) > g(x)$. Then $\min(f, g)(x) = g(x)$.

Let $\varepsilon = (f(x) - g(x))/2$. Choose N_1 and N_2 such that $|\varphi_n(x) - f(x)| < \varepsilon$ for every $n \geq N_1$ and $|\psi_n(x) - g(x)| < \varepsilon$. Then for every $n \geq \max(N_1, N_2)$ we have that

$$\psi_n(x) < g(x) + \varepsilon = f(x) - \varepsilon < \varphi_n(x).$$

Thus, $\min(\psi_n(x), \varphi_n(x)) = \psi_n(x)$ for every $n \geq \max(N_1, N_2)$, so

$$\lim_{n\to\infty} \min(\psi_n, \varphi_n)(x) = \lim_{n\to\infty} \psi_n(x) = g(x),$$

as desired.

To finish the problem note that

– each $\min(\varphi_n, \psi_n)$ is a step function for each n by part (i,

– $\min(\varphi_n, \psi_n)$ is non-negative because φ_n and ψ_n are non-negative and

– the set $A \cup B$ is measure zero because A and B are measure zero.

Problem 4. Let f be an *increasing* function on the closed interval $[a, b] \in \mathbb{R}$. Prove that f is Riemann integrable.
Solution:
For every $\varepsilon > 0$ we produce step functions φ and ψ on $[a, b]$ such that $\varphi \leq f \leq \psi$ and $\int_R \psi - \int_R \phi < \varepsilon$.
Choose an integer n such that

$$n > \frac{\varepsilon}{(b-a)(f(b) - f(a))}.$$

Partition $[a, b]$ into n intervals $[a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n]$ of equal length (i.e. $a_k = a + \frac{k}{n}(b-a)$).
Let

$$\varphi(x) := \begin{cases} f(a_{k-1}) & \text{if } x \in (a_{k-1}, a_k) \text{ for some } k, \\ f(x) & \text{if } x = a_k \text{ for some } k. \end{cases}$$

and

$$\psi(x) := \begin{cases} f(a_k) & \text{if } x \in (a_{k-1}, a_k) \text{ for some } k, \\ f(x) & \text{if } x = a_k \text{ for some } k. \end{cases}$$

By construction φ and ψ are step functions with partition $\{[a_0, a_1], \ldots, [a_{n-1}, a_n]\}$. Since f is increasing, for every $x \in (a_{k-1}, a_k)$ we have that $f(a_{k-1}) \leq f(x) \leq f(a_k)$, so $\varphi \leq f \leq \psi$.
Moreover,

$$\int_R \varphi = \sum_{k=0}^{n-1} \delta f(a_k) \qquad \text{and} \qquad \int_R \psi = \sum_{k=1}^{n} \delta f(a_k)$$

where $\delta = (b-a)/n$ is the length of each of the intervals of the partition $\{[a_0, a_1], \ldots, [a_{n-1}, a_n]\}$.
Thus,

$$\int_R \psi - \int_R \phi = \delta(f(a_n) - f(a_0)) = \frac{1}{n}(b-a)(f(b) - f(b)) < \varepsilon,$$

as desired.
Problem 5. Define the *Cantor set* $C \subset [0, 1]$ to be the set of real numbers in $[0, 1]$ whose base-3 expansion does not contain 1. That is,

$$C := \left\{ x \in [0, 1] \mid x = \sum_{n=1}^{\infty} \frac{a_i}{3^i} \text{ with each } a_i \in \{0, 2\} \right\}.$$

(i) Show that C is uncountable.

(ii) Show that C has Lebesgue measure zero.

Solution:

(i) Let $\mathcal{S}$ be the set of sequences $\{a_n\}$ with $a_i \in \{0, 1\}$. Define a function $S : \mathcal{S} \to C$ by

$$S(\{a_i\}) = \sum_{n=1}^{\infty} \frac{a_i}{3^i}.$$

By Problem 2 from Homework 2, any real number has at most two base-3 expansions and all but countably many real numbers have exactly one base-3 expansion.

Thus, the restriction of S to $\mathcal{S}\backslash A$ (with A at most countable) is an injection into C. Since $\mathcal{S}$ is uncountable and A is countable, $\mathcal{S}\backslash A$ is uncountable. Since S is an injection on $\mathcal{S}\backslash A$, $S(\mathcal{S}\backslash A)$ is also uncountable. Since C contains an uncountable subset $S(\mathcal{S}\backslash A)$, it is uncountable.

Remark: one can actually show that S is injective, i.e. that we can take $A = \emptyset$.

(ii) Write $C = \bigcap_n C_n$, where C_n is the set of all real numbers in $[0, 1]$ that have a base-3 expansion which contains no 1's among the first n digits. *(Note: a number in C_n may have a different base-3 expansion which does contain 1 among its first n digits: for example, $1 = 1.0000$ is an element of every C_n because $1 = 0.22222$.)*

Note that the first n base-3 digits of a number $x \in [0, 1]$ are $0.a_1 \dots a_n$ if and only if $x \in [0.a_1 \dots a_n, 0.a_1 \dots a_n + 1/3^n]$ or equivalently $x \in [\frac{m}{3^n}, \frac{m+1}{3^n}]$ where $m \in [0, 3^n)$ is an integer whose base-3 expansion has no 1's. There are exactly 2^n such integers (we have 2 choices for each digit: 0 and 2). Thus, the total lengths of the intervals comprising C_n is $2^n/3^n$. Since $\lim_{n\to\infty} 2^n/3^n = 0$, we get that C is measure zero.

Problem 6. Show that if $\{I_j\}_{j\in\mathbb{N}}$ is a collection of open intervals in $\mathbb{R}$ which covers $[0, 1]$, meaning that $[0, 1] \in \bigcap_{j=1}^{\infty} I_j$ then $\sum_{j=1}^{\infty} |I_j| \geq 1$. Deduce that $[0, 1]$ does *not* have Lebesgue measure zero. **Hint:** use compactness.

Solution:

We firstly prove the statement for a *finite* collection $\{I_j = (a_j, b_j) \mid 1 \leq j \leq n\}$ of open intervals by induction on n.

For $n = 1$, then we have a single interval (a_1, b_1) covering $[0, 1]$. Hence, $a_1 < 0$ and $b_1 > 1$, so

$$|I_1| = b_1 - a_1 > 1.$$

Assume the induction hypothesis holds for $n - 1$ with $n \geq 2$. We will prove the hypothesis holds for n. Let $\{I_j = (a_j, b_j) \mid 1 \leq j \leq n\}$ be a collection of open intervals covering $[0, 1]$. Pick a k such that $I_k = (a_k, b_k)$ contains 0. Then $a_k < 0 < b_k$. If $b_k > 1$ then I_k covers $[0, 1]$, so $|I_k| > 1$. Assume that $b_k \leq 1$. Then $b_k \in (a_l, b_l)$ for some l.

Consider the interval $I' = (a_k, b_l)$. We have that $I_l \subset I'$ and $I_j \subset I'$, hence $I_l \cup I_k \subset I'$.

The collection of $n - 1$ intervals

$$\{I_j \mid j \neq k, j \neq l\} \cup \{I'\}$$

covers $[0, 1]$, so by the induction assumption,

$$\sum_{j\neq k,l} |I_j| + |I'| \geq 1.$$

Since $b_k \in (a_k, b_k)$ we have that

$$|I_k| + |I_l| = (b_k - a_k) + (b_l - a_l) = (b_l - a_k) + (b_k - a_l) \geq b_l - a_k = |I'|.$$

Thus,

$$\sum_j |I_j| = \sum_{j \neq k,l} |I_j| + |I_k| + |I_l| \geq \sum_{j \neq k,l} |I_j| + |I'| \geq 1,$$

proving the induction hypothesis for n.

Now we are ready to tackle the infinite covers. Let $\{I_j\}_{j \in \mathbb{N}}$ be a collection of intervals covering $[0, 1]$. Since $[0, 1]$ is compact, there exists a finite subcollection $\{I_j\}_{j \in \mathcal{F}}$ that still covers $[0, 1]$. Then, using the result for finite covers,

$$\sum_{j=1}^{\infty} |I_j| \geq \sum_{j \in \mathcal{F}} |I_j| \geq 1.$$