Math 171 Midterm Exam

Wednesday, April 27, 2016, 8:30am - 10:20 am.

Instructions. Answer the following problems carefully and completely. You must show all of your work, stating any result that you are using (unless it is otherwise clear), in order to receive full credit. You are welcome to use all results from the book or class unless otherwise stated, though no references, paper or digital, are permitted. Write your solutions in the provided blue books. Please return this examination, along with any scratch paper used, with your solution books.

If on a multi-part problem you cannot do a part (e.g., part (a)), you may still assume it is true for subsequent parts (e.g., part (b) or (c)) if it is helpful.

There are 5 problems, worth a total of 100 points. You will have 1 hour and 50 minutes to complete the exam.

Name:

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Stanford ID number:

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Signature acknowledging the honor code:

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1. (28 points total, 7 points each) *Prove or disprove.* For each of the following statements, say whether the statement is True or False. Then, prove the statement if it is true, or disprove (find a counterexample with justification) if it is false. (Note: simply stating "True" or "False" will receive no credit).

(a) There exist real numbers which are not rational multiples of the square root of any natural number. That is, the set $S = \mathbb{R} \backslash \{x | x = q\sqrt{n}$ for some $q \in \mathbb{Q}$ and $n \in \mathbb{N}\}$ is non-empty.

(b) If $\sum_{i=1}^{\infty} a_i$ is an infinite series with terms a_i non-negative and $\lim_i a_i = 0$, then $\sum_{i=1}^{\infty} a_i$ absolutely converges.

(c) Let X be any set, and (M, d) any metric space, and suppose $f : X \to M$ is a map of sets. Then the function $d' : X \times X \to [0, \infty)$ given by $d'(x, y) := d(f(x), f(y))$ is a metric on X.

(d) Let $\{a^{(k)}\}_{k \in \mathbb{N}}$ be a sequence of points in a metric space M, and $p \in M$ be any point. If the sequence of distances $\{d(a^{(k)}, p)\}_{k \in \mathbb{N}}$ is a *Cauchy sequence* of real numbers in $\mathbb{R}$, then the sequence of points $\{a^{(k)}\}$ is convergent in M.

2. (20 points total)

(a) (6 points) Give the definition of an *open* subset of a metric space M.

(b) (6 points) Show by example that an arbitrary intersection of open sets need not remain open.

(c) (8 points) Let ℓ^2_3 be the subset of ℓ^2 of sequences $\{a_n\}_{n \in \mathbb{N}}$ such that $a_3 \neq 0$. Show that ℓ^2_3 is an open subset of ℓ^2.

3. (20 points total) Given a pair of sequences of real numbers $\underline{a} := \{a_n\}_{n \in \mathbb{N}}$, $\underline{b} := \{b_n\}_{n \in \mathbb{N}}$, the *splice* of $\underline{a}$, $\underline{b}$ is a new sequence $\underline{c} := \{c_n\}_{n \in \mathbb{N}}$ defined as follows:

$$c_n = \begin{cases} a_{\frac{n+1}{2}} & n \text{ is odd} \\ b_{\frac{n}{2}} & n \text{ is even.} \end{cases}$$

The first few terms of the sequence $\{c_n\}$ are $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$

(a) (12 points) Suppose $\{a_n\}$ and $\{b_n\}$ are both convergent with limits L and M respectively. Calculate (with proof) $\limsup_n c_n$.

(b) (8 points) Suppose as in part (a) that $\lim_n a_n = L$ and $\lim_n b_n = M$. Show that the splice $\{c_n\}$ converges if and only if $L = M$.

4. (20 points total, 10 points each) Let (M, d_M) and (N, d_N) be metric spaces. Recall that $M \times N$ has a natural metric, the *product metric* $d((m, n), (m', n')) = d_M(m, m') + d_N(n, n')$.

(a) Let $p^{(k)} = (m^{(k)}, n^{(k)})$ be a sequence of points in $M \times N$. Show that if $p^{(k)}$ converges if and only if both of the sequences $\{m^{(k)}\}$, $\{n^{(k)}\}$ converge in M and N respectively.

(b) Given a function $f : M \to N$, the *graph of* f is the subset

$$\Gamma_f := \{(x, f(x)) | x \in M\} \subset M \times N.$$

Show that if f is continuous then its graph Γ_f is a *closed subset* of $M \times N$ (equipped with the product metric). **Hint**: At least for one approach, it may be helpful to use the fact that f is continuous if and only if for any $a \in M$, f sends any convergent sequence with limit a to a convergent sequence with limit $f(a)$.

5. (12 points total, 6 points each) Argue with justification whether each of the following sequences absolutely converges, conditionally converges, or diverges.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n (n-2)}{n!}$:

(b) $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{(n+2)(n+3)}{n(n-2)(n-1)}$.