Math 171 Midterm Solutions, Spring 2016

Problem 1.

Prove or disprove. For each of the following statements, say whether the statement is True or False. Then, prove the statement if it is true, or disprove (find a counterexample with justification) if it false. (Note: simply stating "True" or "False" will receive no credit.)

(a) There exist real numbers which are not rational multiples of the square root of any natural number. That is, the set $S = \mathbb{R}\backslash\{x \mid x = q\sqrt{n} \text{ for some } q \in \mathbb{Q} \text{ and } n \in \mathbb{N}\}$ is non-empty.

(b) If $\sum_{i=1}^{\infty} a_i$ is an infinite series with terms a_i non-negative and $\lim_i a_i = 0$ then $\sum_{i=1}^{\infty} a_i$ absolutely converges.

(c) Let X be any set, let (M, d) be any metric space and suppose that $f : X \to M$ is a map of sets. Then the functions $d' : X \times X \to [0, \infty)$ given by $d'(x, y) := d(f(x), f(y))$ is a metric on X.

(d) Let $\{a^{(k)}\}_{k\in\mathbb{N}}$ be a sequence of points in a metric space (M, d) and $p \in M$ be any point. If the sequence of distances $\{d(a^{(k)}, p)\}_{k\in\mathbb{N}}$ is a *Cauchy sequence* of real numbers then the sequence of points $\{a^{(k)}\}$ is convergent in M.

Solution:

(a) **True**. To see this, let P denote the set $\{x \mid x = q\sqrt{n} \text{ for some } q \in \mathbb{Q} \text{ and } n \in \mathbb{N}\}$. Note that one can write $P = \bigcup_{k\in\mathbb{N}} P_k$, where $P_k = \{x \mid x = q\sqrt{k} \text{ for some } q \in \mathbb{Q}\}$. The map $q \mapsto q\sqrt{k}$ gives a bijection $\mathbb{Q} \cong P_k$, so P_k is countable (as $\mathbb{Q}$ is countable by a book result). Thus, P is a countable union of countable sets, hence is countable by a result in the textbook. It follows that while $P \subset \mathbb{R}$, $P \neq \mathbb{R}$, because $\mathbb{R}$ is uncountable by class/the book. It follows that $S = \mathbb{R}\backslash P$ is non-empty.

Note: A popular alternate solution goes as follows: Let x denote any positive irrational number. Then its square root $\sqrt{x}$ exists (though we did not justify this in class!) and cannot be in P, because squares of elements in P are rational. To receive correct points, the student had to justify why the number x chosen was irrational using the methods covered in class. For instance, if the student just picked a number which is "known" to be irrational (but not through our class), such as $\sqrt{2}$ or π, some points were deducted. The easiest way to circumvent this issue is to argue that irrational positive numbers exist by a result in the book (as $(0, \infty)$ is uncountable, but $(0, \infty) \cap \mathbb{Q}$ is countable), and to let x be any such positive irrational number.

(b) **False**. Consider the series $\sum_{i=1} \frac{1}{n}$. Note that the terms are non-negative and $\lim_n \frac{1}{n} = 0$, yet we proved in class/the book that this sum diverges, hence it does not absolutely converge.

(c) **False**. If $f : X \to M$ is not an injective map, then for some $p \in M$ there exist two points x, y with $f(x) = f(y) = p$. It follows that $d'(x, y) = d(f(x), f(y)) = d(p, p) = 0$

even though $x \neq y$, which violates property (i) of a metric. This reasoning, or this reasoning applied to an explicit example of a non-injective map $f : X \to M$ (for example, the map $f : \mathbb{R} \to \mathbb{R}$ sending every point to 0, earned full credit.

(d) **False**. It is true almost by definition that, in a metric space M, if a sequence of points $\{a^{(k)}\}$ converges to $q \in M$, then $\lim_n d(a^{(k)}, q) = 0$, in particular the sequence of real numbers $\{d(a^{(k)}, q)\}$ is Cauchy. However, given some p, if the sequence $\{d(a^{(k)}, p)\}$ is Cauchy (hence by the book convergent to some L), it does *not* follow that $a^{(k)}$ converges (unless of course $L = 0$). Here is a counterexample:

Consider the sequence of points in $\mathbb{R}$: $a^{(k)} = (-1)^k$ in $\mathbb{R}$, and let $p = 0$. Then, note that $b_k := d(a^{(k)}, p) = 1$ for all k, meaning the sequence of distances is the constant sequence $1, 1, \ldots$; one can directly verify this is Cauchy (as $b_m - b_n = 0$ for all m, n, hence is smaller than any $\epsilon > 0$) or cite a result in the book saying that it is convergent hence Cauchy.

However, the sequence of points $a^{(k)}$ does *not* converge in $\mathbb{R}$. To see this, note that $|a^{(k)} - a^{(k-1)}| = 2$ for any k, hence $a^{(k)}$ is not Cauchy (as for any given $\epsilon < 2$, it is impossible to find an N sufficiently large with $|a^{(m)} - a^{(n)}|$, and in particular $|a^{(n+1)} - a^{(n)}|$ less than ϵ for $m, n \geq N$).

Problem 2.

(a) Give the definition of an *open* subset of a metric space M.

(b) Show by example that an arbitrary intersection of open sets need not remain open.

(c) Let ℓ^2_3 be the subset of ℓ^2 of sequences $\{a_n\}$ such that $a_3 \neq 0$. Show that ℓ^2_3 is an open subset of ℓ^2.

Solution:

(a) A subset U of a metric space (M, d) is open if for every $x \in U$ there exists $\varepsilon > 0$ such that the open ball of radius ε around x is contained in U:

$$B_\varepsilon(x) = \{y \in M \mid d(x, y) < \varepsilon\} \subset U.$$

(b) Let $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ because $-\frac{1}{n} < 0 < \frac{1}{n}$ for every positive integer n and for any non-zero real number x, there exists a positive integer n such that $n > \frac{1}{|x|}$, so that for that n, $x \notin U_n$. We have that each U_n is open and $\{0\}$ is *not* open because it does not contain the ε-ball $(-\varepsilon, \varepsilon)$ around 0 for any $\varepsilon > 0$.

Note: many students gave this exact example and wrote that $\{0\}$ is closed, hence it is not open. While $\{0\}$ is indeed closed, a set may be both *open and closed. For example, the whole of $\mathbb{R}$ is both open and closed as a subset of $\mathbb{R}$. Moreover, in a* finite *metric space,* any *subset is both open and closed. And in many metric spaces, there are subsets which are neither open nor closed; consider $(0, 1]$ in $\mathbb{R}$ for instance.*

(c) We will show that the complement $(\ell^2_3)^c$ is closed. Consider a sequence $x^{(k)}$ in $(\ell^2_3)^c$ converging to some $x \in \ell^2$. Then $x^{(k)}_3 = 0$ for every $k \in \mathbb{N}$. Therefore, by the term-wise convergence theorem (for example, see Exercise 37.5) $x_3 = 0$. Thus, $x \in (\ell^2_3)^c$, as desired.

Problem 3.

Given a pair of sequences of real numbers $\underline{a} := \{a_n\}_{n\in\mathbb{N}}$, $\underline{b} := \{b_n\}_{n\in\mathbb{N}}$, the *splice* of $\underline{a}$ and $\underline{b}$ is a new sequence $\underline{c} := \{c_n\}_{n\in\mathbb{N}}$ defined as follows:

$$c_n := \begin{cases} a_{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \\ b_{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

The first few terms of the sequence $\{c_n\}$ are $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$

(a) Suppose $\{a_n\}$ and $\{b_n\}$ are both convergent with limits L and M respectively. Calculate (with proof) $\limsup_n c_n$.

(b) Suppose as in part (a) that $\lim_n a_n = L$ and $\lim_n b_n = M$. Show that the slice $\{c_n\}$ converges if and only if $L = M$.

Solution:

(a)

Lemma 1. *Every convergent subsequence $\{c_{n_k}\}$ of $\{c_n\}$ converges to either L or M.*

Since $\{a_n\}$ and $\{b_n\}$ is bounded, so is $\{c_n\}$, hence $\limsup c_n$ is a real number. Assuming the lemma, we get that

$$\limsup c_n = \sup\{L, M\} = \max(L, M).$$

Proof of Lemma 1. Let K be the limit of the subsequence $\{c_{n_k}\}$. We consider two cases:

- Case 1: there are infinitely many odd integers in $\{n_k\}_{k\in\mathbb{N}}$, i.e. there are infinitely many a_n terms in $\{c_{n_k}\}$. In this case we can consider the subsequence

  $$\{c_{n_k} \mid n_k \text{ is odd}\} = \{a_{\frac{n_k+1}{2}} \mid n_k \text{ is odd}\}$$

  of $\{c_{n_k}\}$ with n_k odd. On one hand it is a subsequence of $\{c_{n_k}\}$, so it converges to K. On the other hand it is a subsequence of $\{a_n\}$, so it converges to L. Thus, $K = L$.

- Case 2: there are only finitely many odd integers in $\{n_k\}_{k\in\mathbb{N}}$, i.e. there are only finitely many a_n terms in $\{c_{n_k}\}$. In this case, there are infinitely many even integers in $\{n_k\}_{k\in\mathbb{N}}$, i.e. there are infinitely many b_n terms in $\{c_{n_k}\}$.

  So we can consider the subsequence

  $$\{c_{n_k} \mid n_k \text{ is even}\} = \{a_{\frac{n_k}{2}} \mid n_k \text{ is even}\}$$

  of $\{c_{n_k}\}$ with n_k even. On one hand it is a subsequence of $\{c_{n_k}\}$, so it converges to K. On the other hand it is a subsequence of $\{b_n\}$, so it converges to M. Thus, $K = M$.

$\square$

(b) In part (a) we showed that the set of limits of converging subsequences of $\{c_n\}$ is $\{L, M\}$. Hence, $\liminf_n c_n = \min\{L, M\}$ and $\limsup c_n = \max\{L, M\}$. By Theorem 20.4 in the book (also discussed in class), $\{c_n\}$ converging is equivalent to $\liminf_n c_n = \limsup_n c_n$ which in turn is equivalent to $\min\{L, M\} = \max\{L, M\}$ and to $L = M$.

Problem 4. Let (M, d_M) and (N, d_N) be metric spaces. Recall that $M \times N$ has a natural metric, the *product metric*, given by

$$d((m, n), (m', n')) = d_M(m, m') + d_N(n, n').$$

(a) Let $p^{(k)} = (m^{(k)}, n^{(k)})$ be a sequence of points in $M \times N$. Show that $p^{(k)}$ converges if and only if both of the sequences $\{m^{(k)}\}$ and $\{n^{(k)}\}$ converge in M and N respectively.

(b) Given a function $f : M \to N$, the *graph of* f is the subset

$$\Gamma_f := \{(x, f(x)) \mid x \in M\} \subset M \times N.$$

Show that if f is continuous the its graph Γ_f is a *closed subset* of $M \times N$ (equipped with the product metric). *Hint: At least for one approach, it may be helpful to use the fact that f is continuous if and only if for any $a \in M$, f sends any convergent sequence with limit a to a convergent sequence with limit $f(a)$.*

Solution:

- Suppose first that the sequence $p^{(k)} = (m^{(k)}, n^{(k)})$ converges to some point $p = (m, n)$. Fix some arbitrary $\epsilon > 0$. By definition, there exists an $N' > 0$ such that for $k \geq N'$, $d_{M\times N}(p^{(k)}, (m, n)) < \epsilon$. But $d_{M\times N}(p^{(k)}, (m, n)) = d_M(m^{(k)}, m) + d_N(n^{(k)}, n)$, and each distance function is non-negative, so it follows that for this same N' and for any $k \geq N'$,

$$d_M(m^{(k)}, m) < \epsilon$$
$$d_N(n^{(k)}, n) < \epsilon.$$

Since we could find such an N' for any given ϵ, it follows that the sequences $\{m^{(k)}\}$ and $\{n^{(k)}\}$ converge to m and n respectively.

Going in the opposite direction, suppose that the sequences $\{m^{(k)}\}$ and $\{n^{(k)}\}$ converge to m and n respectively, and fix some arbitrary $\epsilon > 0$. By definition of convergence of $\{m^{(k)}\}$ to m, there exists an $N_1 > 0$ such that for $k \geq N_1$,

$$d_M(m^{(k)}, m) < \frac{\epsilon}{2}.$$

Similarly, by definition of convergence of $\{n^{(k)}\}$ to n, there exists an $N_2 > 0$ such that for $k \geq N_2$,

$$d_N(n^{(k)}, n) < \frac{\epsilon}{2}.$$

Setting $N' =:= \max(N_1, N_2)$, it follows that for $k \geq N'$,

$$d_{M\times N}(p^{(k)}, (m,n)) = d_M(m^{(k)}, m) + d_N(n^{(k)}, n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

hence since we could find an N' for any given $\epsilon > 0$, it follows that $\{p^{(k)}\}$ converges to (m,n).

(b) By definition, set X in a metric space P is closed if it contains all of its limit points (which implies that the set of limit points of X, $\bar{X}$, is equal to X).

Now, let $p = (m,n) \in M \times N$ be a limit point of the graph Γ_f. By definition, this means that there exists a sequence of points $a^{(k)} \in \Gamma_f$ converging to p. Since $a^{(k)} \in \Gamma_f$, by definition $a^{(k)} = (x^{(k)}, f(x^{(k)})$ for some $x^{(k)} \in M$. By part (a), if a sequence $\{(x^{(k)}, f(x^{(k)})\}$ is convergent in $M \times N$ with limit (m,n), then $x^{(k)}$ is convergent in M with limit m and $f(x^{(k)})$ is convergent in N with limit n. Since f is continuous, the fact that $x^{(k)}$ is convergent with limit m implies that $f(x^{(k)})$ is convergent with limit $f(m)$. By the uniqueness of limits we conclude $n = f(m)$, and hence our limit point $(m,n) = (m, f(m)) \in \Gamma_f$, as desired.

Problem 5. Argue with justification whether each of the following sequences absolutely converges, conditionally converges or diverges.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n(n-2)}{n!}$.

(b) $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{(n+2)(n+3)}{n(n-2)(n-1)}$.

Solution:

(a) The series converges absolutely.

We use the ratio test to show it. We have that the ratio of the adjacent summands equals

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}(n-1)}{(n+1)!} \cdot \frac{n!}{2^n(n-2)} = \frac{2(n-1)}{(n+1)(n-2)}.$$

Hence,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2(n-1)}{(n+1)(n-2)} = \lim_{n\to\infty} \frac{1}{n+1} \cdot 2 \cdot \left(1 + \frac{1}{n-2}\right) = 0 \cdot 2 \cdot 1 = 0.$$

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges absolutely.

(b) The series converges conditionally.

We show that the absolute value series $\sum_{n=3}^{\infty} \frac{(n+2)(n+3)}{n(n-2)(n-1)}$ diverges using comparison test. We note that $(n+2) > (n-2)$ and $(n+3) > (n-1)$, so

$$\frac{(n+2)(n+3)}{n(n-2)(n-1)} > \frac{1}{n},$$

for $n \geq 3$ and we know that $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges.

We show that the series converges conditionally using the Alternating Series Test. We can write

$$\frac{(n+2)(n+3)}{n(n-2)(n-1)} = \frac{1}{n} \cdot \frac{n+2}{n-2} \cdot \frac{n+3}{n-1} = \frac{1}{n} \cdot \left(1 + \frac{4}{n-2}\right) \cdot \left(1 + \frac{4}{n-1}\right).$$

Each of the terms $\frac{1}{n}$, $\left(1 + \frac{4}{n-2}\right)$ and $\left(1 + \frac{4}{n-1}\right)$ is a decreasing function of n, hence so is their product. Also,

$$\lim_{n\to\infty} \frac{(n+2)(n+3)}{n(n-2)(n-1)} = \lim_{n\to\infty} \frac{1}{n} \cdot \left(1 + \frac{4}{n-2}\right) \cdot \left(1 + \frac{4}{n-1}\right) = 0 \cdot 1 \cdot 1 = 0.$$