Mathematics Department, Stanford University
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Preliminary Notes

EXCEPT POSSIBLY FOR THEOREM 2.4 AND THE MATERIAL IN §6 ON COUNTABLE AND UNCOUNTABLE SETS, MOST OF THE MATERIAL HERE—WHICH CORRESPONDS ROUGHLY TO CH.I–VI OF THE TEXT—WILL ALREADY BE FAMILIAR TO YOU, AND WE WILL ONLY SPEND THE FIRST $1\frac{1}{2}$ WEEKS ON IT BEFORE MOVING ON TO CH.VII OF THE TEXT.

Contents

1. THE INTEGERS AND THE REAL NUMBERS

We here assume without discussion the usual properties of the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$; in particular we shall often use the principle of mathematical induction for the positive integers $\mathbb{N} = \{1, 2, \ldots\}$, which is one of the basic properties of the integers. The principle of mathematical induction says that if a subset $\mathcal{P} \subset \mathbb{N}$ has the properties (a) $1 \in \mathcal{P}$, and (b) $n \in \mathcal{P} \Rightarrow n+1 \in \mathcal{P}$ for each $n = 1, 2, \ldots$, then $\mathcal{P} = \mathbb{N}$. Slightly more concretely, that is the same as saying that if P_n is a (true or false) proposition for each $n = 1, 2, \ldots$ and if (a) P_1 is true, and (b) for each n we are able to check that P_{n+1} is true whenever P_n is true, then P_n is true for all $n = 1, 2, \ldots$. Using this principle one can conveniently check many of the other properties of the integers, such as the fact that 1 is the least positive integer and also the "well ordering principle" that any non-empty set of positive integers has a least element. (See Ex.1.1 below.)

We here start with a brief discussion the usual field, order, and supremum axioms for the real numbers; later in the course we shall actually show that the reals can be constructed, starting with the integers, and that these "axioms" become theorems rather than unproven assumptions.

In any case for the moment we accept without discussion that the reals satisfy the field axioms:

F1 $\quad \forall\, a, b : \ a + b = b + a$ and $ab = ba$ (commutativity)

F2 $\quad \forall\, a, b, c : \ a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$ (associativity)

F3 $\quad \forall\, a, b, c : \ a(b + c) = ab + ac$ (distributive law)

F4 $\quad \exists$ elements $0, 1$ with $0 \neq 1$ such that

$0 + a = a$ and $1.a = a \ \forall a$ (additive and multiplicative identities)

F5 $\forall a\ \exists$ an element $-a$ such that $a + (-a) = 0$ (additive inverse)

F6 $\forall a \neq 0\ \exists$ an element a^{-1} such that $aa^{-1} = 1$ (multiplicative inverse)

Notation: $a + (-b)$ and ab^{-1} are usually written $a - b$, a/b respectively. (The latter makes sense only for $b \neq 0$.)

Notice that all the other standard algebraic properties of the reals follow from these. (See Ex. 1.2 below.)

A field (i.e. a set with operations of addition and multiplication satisfying F1–F6) is said to form an "ordered field" if in addition to **F1–F6**, there is a subset P such that the following "order axioms" hold:

O1 $\forall\, a,$ one and only one of the 3 possibilities $a \in P$, $-a \in P, a = 0$ holds

O2 $a,\, b \in P \Rightarrow ab,\, a + b \in P.$

In any ordered field we introduce a relation "$>$", according to the definition $a > b \iff a - b \in P$. Notice that in particular then $a \in P \iff a > 0$, so that, in terms of the relation $>$, P is given by $P = \{a \in \mathbb{R} : a > 0\}$.

Notice this is trivially the case with the reals if we take P to be the set of positive real numbers (so $a > 0$ has its usual meaning in this case); one can then check the other standard properties of inequalities follow from the axioms **O1, O2** and from **F1–F6**, provided $a \geq b$ is formally defined to mean that either $a > b$ or $a = b$ and of course we take $a < b$ to mean $b > a$ and $a \leq b$ to mean $b \geq a$. For example, just using **F1–F6** and the two properties **O1,O2** of the positive reals, we can easily check the standard properties of inequalities listed in Ex. 1.4 below.

Notice also that the above properties (i.e. **F1–F6, O1,O2**) all hold for the rational numbers $\mathbb{Q} = \{p/q : p, q$ are integers with $q \neq 0\}$, with $P = \{p/q : p, q$ are positive integers$\}$.

F1-F6 also hold for the complex numbers $\mathbb{C}$, but $\mathbb{C}$ is <u>not</u> an ordered field; that is, it is impossible to find a subset $P \subset \mathbb{C}$ such that **O1, O2** hold. (See Ex. 1.3 below.)

In addition to **F1–F6, O1,O2** there is one further key property of the real numbers—the "supremum property." To discuss it we need first to introduce some terminology.

Terminology: If $S \subset \mathbb{R}$, we say:

(1) S is <u>bounded above</u> if $\exists$ a real number K such that $x \leq K\ \forall x \in S$. (Any such number K is called an upper bound for S.)

(2) S is <u>bounded below</u> if $\exists$ a number k such that $k \leq x\, \forall x \in S$. (Any such number k is called a lower bound for S.)

(3) S is <u>bounded</u> if it is both bounded above and bounded below. (This is equivalent to the fact that $\exists$ a real number L such that $|x| \leq L\, \forall x \in S$.)

We can now introduce the supremum property of the real numbers, which is the final axiom:

S. (Least upper bound or "supremum" property of the reals): If $S \subset \mathbb{R}$ is non-empty and bounded above, then S has a least upper bound.

Notice that the terminology "least upper bound" used here means exactly what it says; a number α is a least upper bound for a set $S \subset \mathbb{R}$ if

(i) $x \le \alpha \ \forall x \in S$ (i.e. α is an upper bound), and

(ii) $\alpha \le \beta$ for every upper bound β of S. (i.e. α is $\le$ any other upper bound for S.)

Notice that such a least upper bound is unique, because if α_1, α_2 are both least upper bounds for S, then property (ii) implies that both $\alpha_1 \le \alpha_2$ and $\alpha_2 \le \alpha_1$, hence $\alpha_1 = \alpha_2$. It therefore makes sense to speak of the least upper bound of S, (also known as "the supremum" of S).

Notation: The least upper bound of S will henceforth be denoted $\sup S$ or alternatively $\operatorname{lub} S$. With this notation, property **S** says that $\sup S$ exists if S is non-empty and bounded above.

Remark: If S is non-empty and bounded below, then it has a greatest lower bound (or "infimum"), which we denote $\inf S$, or, alternatively, $\operatorname{glb} S$. One can prove the existence of $\inf S$ (if S is bounded below and non-empty) by using property **S** on the set $-S = \{-x : x \in S\}$. (See Ex. 1.6 below.)

We should be careful to distinguish between the maximum element of a set S (if it exists) and the supremum of S. Recall that we say a number α is the maximum of S (denoted $\max S$) if

(i)′ $x \le \alpha \quad \forall x \in S$, and

(ii)′ $\alpha \in S$.

These two properties say exactly that α is both an upper bound for S and also one of the elements of S. Thus clearly (comparing (i), (ii) and (i)′, (ii)′), we see that, for any non-empty set $S \subset \mathbb{R}$ which is bounded above,

$$(*) \qquad \max S \text{ exists} \iff \sup S \in S,$$

in which case $\max S = \sup S$. In particular (by the direction "$\Rightarrow$" of $(*)$) if $\sup S \notin S$, then $\max S$ does not exist. For example, if $S = (0,1) = \{x : 0 < x < 1\}$, then $\sup S = 1$, but $\max S$ does not exist, because $1 \notin S$. Notice that of course any finite non-empty set has a maximum. (See Ex. 1.9 below.)

The following Archimedean Property of the reals would normally be accepted (and freely used) without proof. Here we are being a little more formal than usual, so we will actually give a formal proof based on mathematical induction and the supremum axiom **S**.

Lemma 1.1 (Archimedean Property of $\mathbb{R}$.) The set $\mathbb{N}$ of all positive integers is not bounded above (hence for each $\epsilon > 0$ there is a positive integer n with $\frac{1}{n} < \epsilon$—i.e. $n > 1/\epsilon$).

Proof. (By contradiction.) If the result is false, then the set $\mathbb{N}$ of positive integers is non-empty (it contains 1) and bounded above, hence by property **S** has a least upper bound α. Thus in particular

$$n \le \alpha \quad \forall \text{ positive integer } n.$$

However if n is a positive integer, then so is $n+1$, hence

$$n+1 \le \alpha \quad \forall \text{ positive integer } n,$$

so that

$$n \le \alpha - 1 \quad \forall \text{ positive integer } n,$$

thus showing that $\alpha - 1$ is an upper bound for $\mathbb{N}$, contradicting the fact that α was chosen to be the least upper bound.

Note: Using the above Archimedean Property we can also establish various other basic properties connecting the reals and the integers. For example for each real number $a \ge 0$ there is a positive integer n such that $n-1 \le a < n$. (The value $n-1$ is sometimes referred to as the "integer part" of a.) To prove this consider the set $S = \{n \in \mathbb{N} \text{ such that } n \le a+1\}$. We claim that there must be an $n_0 \in S$ such that $n_0 + 1 \notin S$—which says $n_0 + 1 > a+1$, or equivalently $n_0 > a$. (Indeed otherwise we would have $n \in S \Rightarrow n+1 \in S$ for each $n \in \mathbb{N}$, and since $1 \in S$ the principle of mathematical induction would then tell us that $S = \mathbb{N}$, which contradicts the above Archimedean Property because S is, by definition, bounded above.) Observe by construction that n_0 then has the property that $n_0 - 1 \le a < n_0$ as required.

Final Note on the Reals:

We here assume without proof all the properties **F1–F6, O1,O2**, and **S**. Actually, as we mentioned above, it is possible, starting only with the positive integers, to give a rigorous construction of the real numbers, and <u>prove</u> all the properties **F1–F6, O1,O2**, and **S**. We shall in fact give this proof later in the course.

It is also possible to prove (in a sense that can be made precise) that the reals are the unique field with all the properties **F1–F6, O1,O2**, and **S**.

Problems

1.1 Prove (a) 1 is the least element of $\mathbb{N}$ (i.e. $1 < n \, \forall n \in \mathbb{N} \setminus \{1\}$), and (b) the "well ordering principle" that every non-empty $T \subset \mathbb{N}$ has a least element (i.e. there is $m \in T$ such that $m < n \, \forall n \in T \setminus \{m\}$).

Note: We later want to argue that we can <u>construct</u> the reals starting only with the integers, and so it is best to here assume only what the axioms of set theory initially give us about the integers: that there is a set $\mathbb{N}$ (called the "positive integers") with a total ordering denoted "$<$" (thus (i) for each m, n with $n \ne m$ exactly one of $n < m$ or $m < n$ holds, and (ii) $n < n$ holds for no n, and (iii) $\forall \ell, n, m$, $\ell < m$ and $m < n \Rightarrow \ell < n$), $\exists$ an element $1 \in \mathbb{N}$ (called "one"), such that corresponding to each $n \in \mathbb{N}$ there is another element, denoted $n+1$ (called the "successor of n"), such that the "principle of mathematical induction" is true—i.e. ($S \subset \mathbb{N}$ with $1 \in S$ and $n \in S \Rightarrow n+1 \in S \, \forall n \in \mathbb{N}$) $\Rightarrow S = \mathbb{N}$, and the successor $n+1$ has the additional properties that, for all $n \in \mathbb{N}$, $n < n+1$ and $n < m \Rightarrow n+1 \le m$.[1]

Hint for (a): Let $S = \{n \in \mathbb{N} : 1 < n\}$, and prove that $S \cup \{1\} = \mathbb{N}$.

Hint for (b): Let $T \subset \mathbb{N}$ be non-empty. We can assume that $1 \notin T$ otherwise by (a) 1 is the least element of T and there is nothing further to prove. Let $S = \{m \in \mathbb{N} : m < n \, \forall \, n \in T\}$ and note in particular that $1 \in S$ by (a) and the fact that $1 \notin T$. Note that either $m \in S \Rightarrow m+1 \in S$, or else there is an $m_0 \in S$ with $m_0 + 1 \notin S$. In the latter case you should argue (using the properties in the above note) that $m_0 + 1 \in T$ and is in fact the required least element of T.

1.2 Presenting your argument in steps, using only properties **F1–F6** and stating which of **F1-F6** is used at each step, prove

(i) $\forall a \in \mathbb{R},\ a.0 = 0.$
(ii) $\forall a, b \in \mathbb{R}, \quad ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$
(iii) $\frac{a}{b} = \frac{ac}{bc}\ \forall a, b, c \in \mathbb{R}$ with $b \neq 0$ and $c \neq 0$
(iv) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}\ \forall a, b, c, d$ with $b \neq 0$ and $d \neq 0$.

1.3 (i) If F is any ordered field with multiplicative and additive identities denoted as usual by 1,0, prove that $1 > 0$.

(ii) Prove that $\mathbb{C}$ (the complex numbers) do not form an ordered field; that is, there is no subset $P \subset \mathbb{C}$ such that axioms **O1, O2** hold.

1.4 Presenting your argument in steps, using only properties **F1–F6, O1,O2** and stating which of **F1–F6, O1,O2** is used at each step, prove the following:

(i) $a > 0 \implies 0 > -a \quad (\text{ i.e. } -a < 0).$
(ii) $a > 0 \implies \frac{1}{a} > 0.$
(iii) $a > b > 0 \implies \frac{1}{a} < \frac{1}{b}.$
(iv) $a > b$ and $c > 0 \implies ac > bc.$
(v) $a > b$ and $c < 0 \implies ac < bc.$

1.5 If $S = \{1 - \frac{1}{n} : n = 1, 2, \ldots\}$, prove carefully that $\sup S = 1$. Note that this means you have to establish properties (i), (ii) (in the definition of $\sup S$) with $\alpha = 1$.

1.6 Given a set $S \subset \mathbb{R}$, $-S$ denotes $\{-x : x \in S\}$. Prove (i) S is bounded below if and only if $-S$ is bounded above, and (ii) If S is non-empty and bounded below, then $\inf S$ exists and is equal to $-\sup(-S)$.

1.7 Prove that every positive real number has a positive square root. (That is, for any $a > 0$, prove there is a real number $\alpha > 0$ such that $\alpha^2 = a$.) (Hint: Let $S = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < a\}$, and begin by showing that S is non-empty and bounded above.)

1.8 Prove that if $a \in \mathbb{R}$ and $\epsilon > 0$ then the interval $(a, a + \epsilon)$ contains both rational and irrational numbers. (Recall x rational means $x = p/q$ with p, q integers and $q \neq 0$; x irrational means x is not rational.)
Hint: Start by selecting an integer $q > 0$ such that $\frac{1}{q} < \epsilon$, so that $(qa, q(a + \epsilon))$ is an interval of length greater than 1; you can of course use the fact that the real numbers can be written as the disjoint union $\cup_{n \in \{0, \pm 1, \pm 2, \ldots,\}}[n, n + 1)$ (which is a restatement of the important general fact every real x can be uniquely written in the form $x = n + y$ with n an integer and $y \in [0, 1)$—such n is referred to as the "integer part of x").

1.9 If S is a finite non-empty subset of $\mathbb{R}$, prove that $\max S$ exists. (Hint: Let n be the number of elements of S and use the order properties **O1,O2** together with induction on n.)

2. SEQUENCES OF REAL NUMBERS

Let $a_1, a_2, \ldots$ be a sequence of real numbers; a_n is called the n-th term of the sequence. We sometimes use the abbreviation $\{a_n\}$ or $\{a_n\}_{n=1,2,\ldots}$ to denote the sequence; notice that your text uses the notation (a_n) rather than $\{a_n\}$.

Technically we should distinguish between the sequence $\{a_n\}$ and the set of terms of the sequence—i.e. the set $S = \{a_n : n = 1, 2, \ldots\}$. These are not the same: e.g. the sequence $1, 1, \ldots$ has infinitely many terms each equal to 1, whereas the set S is just the set $\{1\}$ containing one element.

Formally a sequence is a mapping from the positive integers $\mathbb{N}$ to the real numbers; the n-th term of the sequence is just the value of this mapping at the integer n. From this point of view—i.e. thinking of a sequence as a mapping from the integers to the real numbers—a sequence has a graph consisting of discrete points in $\mathbb{R}^2$, one point of the graph on each of the vertical lines $x = n$. Thus for example the sequence $1, 1, \ldots$ (each term=1) has graph consisting of the discrete points with coordinates $(n, 1)$, $n = 1, 2, \ldots$ (all lying on the line $y = 1$ in $\mathbb{R}^2$).

Terminology: Recall the following terminology.

A sequence $\{a_n\}$ is:

(i) bounded above if $\exists$ a real number K such that $a_n \le K \, \forall n \in \mathbb{N}$.

(ii) bounded below if a real number k such that $a_n \ge k \, \forall n \in \mathbb{N}$.

(iii) bounded if it is both bounded above and bounded below. (This is equivalent to the fact that $\exists$ a real number L such that $|a_n| \le L \, \forall n \in \mathbb{N}$.)

(iv) increasing if $a_{n+1} \ge a_n \, \forall n \in \mathbb{N}$.

(v) strictly increasing if $a_{n+1} > a_n \, \forall n \in \mathbb{N}$

(vi) decreasing if $a_{n+1} \le a_n \, \forall n \in \mathbb{N}$.

(vii) strictly decreasing if $a_{n+1} < a_n \, \forall n \in \mathbb{N}$.

(viii) monotone if either the sequence is increasing or the sequence is decreasing.

Also, the sequence:

(ix) is said to have limit ℓ (ℓ a given real number) if for each $\epsilon > 0$ there is an integer $N \in \mathbb{N}$ such that

$$(*) \qquad |a_n - \ell| < \epsilon \quad \forall \text{ integer } n \ge N.$$

(x) is said to converge if it has limit ℓ for some ℓ. (Of course such limit ℓ, if it exists, is unique.)

(Notice that $(*)$ is equivalent to $\ell - \epsilon < a_n < \ell + \epsilon \quad \forall$ integer $n \ge N$, which in turn is equivalent to $a_n \in (\ell - \epsilon, \ell + \epsilon) \quad \forall n \ge N$.)

In case the sequence $\{a_n\}$ has limit ℓ we write $\lim a_n = \ell$, or $\lim_{n\to\infty} a_n = \ell$ or $a_n \to \ell$.

(Notice that we never write anything like $\lim a_n \to \ell$.)

Theorem 2.1. *If $\{a_n\}$ is monotone and bounded, then it is convergent. In fact if $S = \{a_1, a_2, \ldots\}$ is the set of terms of the sequence, we have the following:*

(i) If $\{a_n\}$ is increasing and bounded then $\lim a_n = \sup S$.

(ii) if $\{a_n\}$ is decreasing and bounded then $\lim a_n = \inf S$.

Proof: See Ex. 2.2. (Ex. 2.2 proves part (i), and the proof of part (ii) is almost identical.)

Theorem 2.2. *If $\{a_n\}$ is convergent, then it is bounded.*

Proof: Let $\ell = \lim a_n$. Using the definition (ix) above with $\epsilon = 1$, we see that there exists an $N \in \mathbb{N}$ such that $|a_n - \ell| < 1$ whenever $n \geq N$. By the triangle inequality we have $|a_n| = |(a_n - \ell) + \ell| \leq |a_n - \ell| + |\ell|$ for each integer $n \in \mathbb{N}$, and hence we have $|a_n| \leq 1 + |\ell|$ for each $n \geq N$. Therefore $|a_n| \leq \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |\ell|\}$ for all $n \in \mathbb{N}$.

Theorem 2.3 (The algebra of limits). *If $\{a_n\}$, $\{b_n\}$ are convergent sequences, then the sequences $\{a_n + b_n\}$, $\{a_n b_n\}$ are also convergent, and*

(i)
$$\lim(a_n + b_n) = \lim a_n + \lim b_n$$

(ii)
$$\lim(a_n b_n) = (\lim a_n)(\lim b_n).$$

In addition, if $b_n \neq 0 \,\forall n \in \mathbb{N}$ and $\lim b_n \neq 0$, then $\lim(a_n/b_n)$ exists, and

(iii)
$$\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}.$$

Proof: Exercise in the use of definition (ix) above.

Remark: Notice that if we take $\{a_n\}$ to be the constant sequence $-1, -1, \ldots$ (so that trivially $\lim a_n = -1$ in part(ii) above), we get $\lim(-b_n) = -\lim b_n$, and hence using part (i) with $\{-b_n\}$ in place of $\{b_n\}$ we conclude that

$$\lim(a_n - b_n) = \lim a_n - \lim b_n,$$

provided both $\lim a_n$ and $\lim b_n$ exist. By a similar argument (i), (ii) imply that

$$\lim(\alpha a_n + \beta b_n) = \alpha \lim a_n + \beta \lim b_n$$

for any α, $\beta \in \mathbb{R}$, provided both $\lim a_n$ and $\lim b_n$ exist.

The next theorem, known as the **Bolzano-Weierstrass Theorem** is of central importance in analysis:

2.4 Theorem. *Every bounded sequence of real numbers has a convergent subsequence.*

Note: A <u>subsequence</u> of $\{a_n\}_{n=1,2,\ldots}$ is any sequence of the form $\{a_{n_j}\}_{j=1,2,\ldots}$, where $1 \leq n_1 < n_2 < \cdots$. (Notice that then $n_{j+1} \geq n_j + 1 \,\forall\, j \geq 1$, and hence $n_j \geq j \,\forall\, j$.)

Proof of 2.4: Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$, and let K, L be such that $K \leq a_n \leq L$ for all $n \geq 1$. Since $\{a_j : j \geq n\}$ is a non-empty subset of $[K, L]$ for each $n \in \mathbb{N}$, we can define

$$A_n = \sup\{a_j : j \geq n\}.$$

Notice that by definition $A_{n+1} \leq A_n$, and $K \leq A_n$ for each $n \geq 1$. Thus by Theorem 2.1 we know that $\{A_n\}$ is a convergent sequence; let $\ell = \lim A_n$. For each $j = 1, 2, \ldots$, the ϵ, N

definition of limit with $\epsilon = 1/j$ implies $\exists\, N_j$ with $A_n \in (\ell - 1/j, \ell + 1/j)$ for all $n \geq N_j$, so in particular (since $a_n \leq A_n \forall\, n$)

$$a_n < \ell + 1/j \quad \forall\, n \geq N_j. \tag{1}$$

On the other hand we must also have

$$\ell - 1/j < a_n \text{ for infinitely many } n \tag{2}$$

because otherwise $\exists\, N$ such that $a_n \leq \ell - 1/j$ for all $n \geq N$, which by definition of A_n implies $A_n \leq \ell - 1/j$ for all $n \geq N$, contradicting the fact that $\lim A_n = \ell$. Observe that (1),(2) imply that for each $j = 1, 2, \ldots$

$$a_n \in (\ell - 1/j, \ell + 1/j) \text{ for infinitely many } n. \tag{3}$$

Then (by (3) with $j = 1$) we can select $n_1 \geq 1$ with $a_{n_1} \in (\ell - 1, \ell + 1)$, and (by (3) with $j = 2$) we can select $n_2 > n_1$ with $a_{n_2} \in (\ell - 1/2, \ell + 1/2)$, and for any $j \geq 2$ we can use (3) to inductively select $n_j > n_{j-1}$ with $a_{n_j} \in (\ell - 1/j, \ell + 1/j)$. Thus we have constructed a subsequence $\{a_{n_j}\}$ with $\lim a_{n_j} = \ell$.

Important Note: If $\{A_n\}$ is the sequence constructed in the above proof (i.e., $A_n = \sup_{j \geq n} a_j$ for each $n \in \mathbb{N}$), then $\lim A_n$ is usually called the "limit superior" or "lim sup" of the sequence: we write

$$\limsup_{n\to\infty} a_n = \lim A_n \quad \text{or} \quad \overline{\lim}\, a_n = \lim A_n.$$

Likewise (still assuming $\{a_n\}$ is bounded) we define $B_n = \inf_{j \geq n} a_j$ (so that $B_{n+1} \geq B_n$ and $\{B_n\}$ is bounded), and we write

$$\liminf_{n\to\infty} a_n = \lim B_n \quad \text{or} \quad \underline{\lim}_{n\to\infty}\, a_n = \lim B_n.$$

Problems

2.1 Use the Archimedean property of the reals (i.e. Lemma 1.1) to prove rigorously that $\lim 1/n = 0$.

2.2 Prove part (i) of Theorem 2.1 (Hint: Let $\alpha = \sup S$, and show first that for each $\epsilon > 0$ there is an integer $N \in \mathbb{N}$ such that $a_N > \alpha - \epsilon$.)

2.3 Using the definition (ix) above to prove that a sequence $\{a_n\}$ cannot have more than one limit.

2.4 ("Pinching Theorem") If $\{a_n\}$, $\{b_n\}$ are given convergent sequences with $\lim a_n = \lim b_n$, and if $\{c_n\}$ is any sequence such that $a_n \leq c_n \leq b_n \forall n \geq 1$, prove that $\{c_n\}$ is convergent and $\lim a_n = \lim b_n = \lim c_n$.

2.5 Use the algebra of limits (i.e. Theorem 2.3 and the remark following it) to rigorously justify the statement $\lim((n^3 - 6)/(n^3 + n^2 + 1)) = 1$. (Of course you can assume without discussion that $\lim 1/n = 0$.)

2.6 If $\{a_n\}$ is a bounded sequence, prove that

$$\underline{\lim}_{n\to\infty}\, a_n = \overline{\lim}_{n\to\infty}\, a_n \iff \{a_n\} \text{ converges.}$$

2.7 If $\{a_n\}$ is bounded, prove that

$$\limsup_{n\to\infty} a_n = \sup\{\ell : \ell \text{ is the limit of some subsequence of } \{a_n\}\}$$

3. SERIES OF REAL NUMBERS.

Consider the series

$$a_1 + a_2 + \cdots + a_n + \cdots$$

(usually written with summation notation as $\sum_{n=1}^{\infty} a_n$), where $a_1, a_2, \ldots$ is a given sequence of real numbers. a_n is called the $\underline{n^{\text{th}}\text{ term}}$ of the series. The sum of the first n-terms, i.e. $s_n = \sum_{k=1}^{n} a_k$, is called the $\underline{n^{\text{th}}\text{ partial sum}}$ of the series.

If $s_n \to s$ (i.e. if $\lim s_n = s$) for some $s \in \mathbb{R}$, then we say the series converges, and has sum s. Also, in this case we write $\sum_{n=1}^{\infty} a_n = s$.

If $\{s_n\}_{n=1,2,\ldots}$ does not converge, then we say the series diverges.

Example: If $a \in \mathbb{R}$ is given, then the series $1 + a + a^2 + \cdots$ (i.e. the geometric series with common ratio a) has n^{th} partial sum

$$\begin{aligned} s_n &= 1 + a + \cdots + a^{n-1} \\ &= \begin{cases} n & \text{if } a = 1 \\ \dfrac{1-a^n}{1-a} & \text{if } a \neq 1 \end{cases} \end{aligned}$$

Using the fact that $a^n \to 0$ if $|a| < 1$, we thus see that the series converges and has sum $1/(1-a)$ if $|a| < 1$, whereas the series diverges for $|a| \geq 1$. (Indeed $\{s_n\}$ is unbounded if $|a| > 1$ or $a = 1$, and, if $a = -1$, $\{s_n\} = 1, 0, 1, 0, \ldots$)

The following simple lemma is of key importance.

Lemma 3.1. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim a_n = 0$.*

Note: The converse is not true. For example, we check below that $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, but its n-th term is $1/n$, which does converge to zero.

Proof of Lemma 3.1 Let $s = \lim s_n$. Then of course we also have $s = \lim s_{n+1}$, and hence $\lim(s_{n+1} - s_n) = 0 - 0 = 0$ (by Theorem 2.3). But, for each $n \geq 1$, $s_{n+1} - s_n = a_{n+1}$, so we have shown $\lim a_{n+1} = 0$; that is, $\lim a_n = 0$. (Notice that we here use the fact that, for any $\ell \in \mathbb{R}$, $\lim a_{n+1} = \ell \iff \lim a_n = \ell$; this is trivially true because $\{a_{n+1}\}_{n=1,2,\ldots} = \{a_n\}_{n=2,3,\ldots}$—i.e. the sequence $\{a_n\}_{n=1,2,\ldots}$ with the first term deleted.)

The following lemma is of both theoretical and practical importance.

Lemma 3.2. *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, and have sums s, t respectively, and if α, β are real numbers, then $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ also converges, and has sum $\alpha s + \beta t$. (i.e. $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$ if both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.)*

Proof: Let $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$. We are given $s_n \to s$ and $t_n \to t$. Then $\alpha s_n + \beta t_n \to \alpha s + \beta t$. (See the remarks following Theorem 2.3.) But now $\alpha s_n + \beta t_n = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(\alpha a_k + \beta b_k)$, which is the n^{th} partial sum of $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$.

There is a very convenient criterion for checking convergence in case all the terms are non-negative. Indeed in this case

$$s_{n+1} - s_n = a_{n+1} \geq 0 \quad \forall n \geq 1,$$

hence the sequence $\{s_n\}$ is <u>increasing</u>. Thus by Theorem 2.2 (i) we see that $\{s_n\}$ converges if and only if it is bounded. That is, we have proved:

Lemma 3.3. *If each term of $\sum_{n=1}^{\infty} a_n$ is non-negative, then the series converges if and only if the sequence of partial sums $\{s_n\}$ is <u>bounded</u>.*

Example: Using the above criterion, we can discuss convergence of $\sum_{n=1}^{\infty} 1/n^p$, where $p > 0$ is given. The n^{th} partial sum is

$$s_n = \sum_{k=1}^{n} \frac{1}{k^p}.$$

Since $1/x^p$ is a decreasing function of x for $x > 0$, we have, for each integer $k \geq 1$,

$$\frac{1}{(k+1)^p} \leq \frac{1}{x^p} \leq \frac{1}{k^p} \quad \forall x \in [k, k+1].$$

Integrating, this gives

$$\frac{1}{(k+1)^p} = \int_k^{k+1} \frac{1}{(k+1)^p}\, dx \leq \int_k^{k+1} \frac{1}{x^p}\, dx \leq \int_k^{k+1} \frac{1}{k^p}\, dx = \frac{1}{k^p},$$

so if we sum from $k = 1$ to n, we get

$$\sum_{k=1}^{n} \frac{1}{(k+1)^p} \leq \int_1^{n+1} \frac{1}{x^p}\, dx \leq \sum_{k=1}^{n} \frac{1}{k^p}.$$

That is,

$$s_{n+1} - 1 \leq \int_1^{n+1} \frac{1}{x^p} \leq s_n \quad \forall n \geq 1.$$

But

$$\int_1^{n+1} \frac{1}{x^p} = \begin{cases} \log(n+1) & \text{if } p = 1 \\ \dfrac{(n+1)^{1-p} - 1}{1-p} & \text{if } p \neq 1\,, \end{cases}$$

and thus we see that $\{s_n\}$ is <u>unbounded</u> if $p \leq 1$ and <u>bounded</u> if $p > 1$. Hence from Lemma 3.2 we conclude that $\sum_{n=1}^{\infty} 1/n^p$ converges for $p > 1$ and diverges for $p \leq 1$.

Remark: The method used in the above proof can be modified to discuss convergence of a large class of series—see ex. 3.4 below.

Theorem 3.4. *If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Terminology: If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u>. Thus, with this terminology, the above theorem just says "absolute convergence implies convergence."

Proof of Theorem 3.4: Let $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} |a_k|$. Then we are given $t_n \to t$ for some $t \in \mathbb{R}$.

For each integer $n \geq 1$, let

$$p_n = \begin{cases} a_n \text{ if } a_n \geq 0 \\ 0 \text{ if } a_n < 0 \end{cases} \qquad q_n \quad = \begin{cases} -a_n \text{ if } a_n \leq 0 \\ 0 \text{ if } a_n > 0 \end{cases},$$

and let $s_n^+ = \sum_{k=1}^n p_n$, $s_n^- = \sum_{k=1}^n q_n$. Notice that then for each $n \geq 1$ we have

$$\begin{aligned} a_n &= p_n - q_n, \quad s_n = s_n^+ - s_n^- \\ |a_n| &= p_n + q_n, \quad t_n = s_n^+ + s_n^-, \end{aligned}$$

and $p_n, q_n \geq 0$. Notice that then

$$0 \leq s_n^+ \leq t_n \leq t, \text{ and } 0 \leq s_n^- \leq t_n \leq t$$

for every $n \geq 1$, hence we have shown that $\sum_{n=1}^\infty p_n$ and $\sum_{n=1}^\infty q_n$ have bounded partial sums and hence both converge by Lemma 3.3. But then (by Lemma 3.2) $\sum_{n=1}^\infty (p_n - q_n)$ converges; that is, $\sum_{n=1}^\infty a_n$ converges as required.

We next want to show that the terms of an absolutely convergent series can be rearranged in an arbitrary way without changing the sum. First we make the definition clear.

Definition: Let $j_1, j_2, \ldots$ be any sequence of positive integers in which every positive integer appears once and only once. (i.e. the mapping $n \mapsto j_n$ is a $1:1$ mapping of $\mathbb{N}$ onto $\mathbb{N}$.) Then the series $\sum_{n=1}^\infty a_{j_n}$ is said to be a rearrangement of the series $\sum_{n=1}^\infty a_n$.

Theorem 3.5. *If $\sum_{n=1}^\infty a_n$ is absolutely convergent, then any rearrangement $\sum_{n=1}^\infty a_{j_n}$ converges absolutely, and has the same sum as $\sum_{n=1}^\infty a_n$.*

Proof: We give the proof when $a_n \geq 0$ (in which case "absolute convergence" just means "convergence"). The extension to the general case is left as an exercise. (See Ex. 3.6 below.) Hence assume $\sum_{n=1}^\infty a_n$ converges, and $a_n \geq 0\, \forall n \geq 1$, and let $\sum_{n=1}^\infty a_{j_n}$ be any rearrangement. For each $n \geq 1$, let

$$P(n) = \max\{j_1, \ldots, j_n\},$$

so that

$$\{j_1, \ldots, j_n\} \subset \{1, \ldots, P(n)\},$$

and hence (since $a_k \geq 0; \forall k \geq 1$)

$$a_{j_1} + a_{j_2} + \cdots + a_{j_n} \leq a_1 + a_2 + \cdots + a_{P(n)} \leq s,$$

where $s = \sum_{n=1}^\infty a_n$. Thus we have shown that the partial sums of $\sum_{n=1}^\infty a_{j_n}$ are bounded above by s, hence by Lemma 3.3, $\sum_{n=1}^\infty a_{j_n}$ converges, and has sum t satisfying $t \leq s$. But now $\sum_{n=1}^\infty a_n$ is a rearrangement of $\sum_{n=1}^\infty a_{j_n}$ (using the rearrangement given by the inverse mapping $j_n \mapsto n$, $n \in \mathbb{N}$), and hence by the same argument we also have $s \leq t$. Hence $s = t$ as required.

Problems.

3.1 (i) (Comparison test absolute convergence.) If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are given series and if $|a_n| \le |b_n|\ \forall n \in \mathbb{N}$, prove that $\sum_{n=1}^{\infty} b_n$ absolutely convergent $\implies$ $\sum_{n=1}^{\infty} a_n$ absolutely convergent.

(ii) Use this test to discuss convergence of

(a) $\sum_{n=1}^{\infty} (\sin n)/n^2$

(b) $\sum_{n=1}^{\infty} (\sin(1/n))/n$.

3.2 (Comparison test for divergence.) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are given series with non-negative terms, and if $a_n \ge b_n\ \forall n \in \mathbb{N}$, prove that $\sum_{n=1}^{\infty} b_n$ diverges $\implies$ $\sum_{n=1}^{\infty} a_n$ diverges.

3.3 Suppose $a_n \ge 0\ \forall n \in \mathbb{N}$. Prove

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges if $\sum_{n=1}^{\infty} a_n$ diverges.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$ converges, whether or not $\sum_{n=1}^{\infty} a_n$ converges.

3.4 (Integral test.) If $f : [1,\infty) \to \mathbb{R}$ is positive and continuous at each point of $[1,\infty)$, and if f is decreasing, prove, using a modification of the argument used to discuss convergence of $\sum_{n=1}^{\infty} 1/n^p$ above, that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\{\int_1^n f(x)\,dx\}_{n=1,2,\ldots}$ is a bounded sequence.

3.5 Use the integral test (in Ex. 3.4 above) to discuss convergence of

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\log(n+1)}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\log(n+1))^2}$.

3.6 Complete the proof of Theorem 3.5 (i.e. discuss the general case when $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.)

Hint: The theorem has already been established for series of non-negative terms; use p_n, q_n as in Theorem 3.4.

4. POWER SERIES.

A power series is a series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

(in summation notation $\sum_{n=0}^{\infty} a_nx^n$), where $a_0, a_1, \ldots$ are given real numbers and x is a real variable. Notice that we also here use the notational convention that $x^0 = 1$ for all x.

Observe that for $x = 0$ the series trivially converges and its sum is a_0.

The following theorem describes the key convergence property of such series.

Theorem 4.1. *For any power series $\sum_{n=0}^{\infty} a_nx^n$, exactly one of the following 3 possibilities holds:*

(i) *the series diverges $\forall x \neq 0$,* **(ii)** *the series converges absolutely $\forall x \in \mathbb{R}$,* **(iii)** *$\exists \rho > 0$ such that the series converges absolutely $\forall x$ with $|x| < \rho$, and diverges $\forall x$ with $|x| > \rho$.*

Terminology: If (iii) holds, the number ρ is called the radius of convergence and the interval $(-\rho, \rho)$ is called the interval of convergence. If (i) holds we say the radius of convergence is zero, and if (ii) holds we say radius of convergence $= \infty$.

Note: The theorem says nothing about what happens at $x = \pm\rho$ in case (iii).

Proof of Theorem 4.1 Suppose the series converges at a point $x_1 \in \mathbb{R} \setminus \{0\}$, and let $|x| < |x_1|$. Then

$$|a_nx^n| = |a_n||x|^n = |a_n||x_1|^n\left|\tfrac{x}{x_1}\right|^n. \tag{1}$$

Now, since $\lim |a_nx_1^n| = 0$ (by Lemma 3.1 above), there is an N such that $|a_n||x_1|^n \leq 1 \,\forall\, n \geq N$, and hence (1) implies that

$$|a_nx^n| \leq \left|\tfrac{x}{x_1}\right|^n, \quad n \geq N.$$

Hence, since the geometric series $\sum_{n=1}^{\infty} |\frac{x}{x_1}|^n$ converges (because $|\frac{x}{x_1}| < 1$), we conclude by the comparison test (Ex. 3.1 above) that

$$\sum_{n=0}^{\infty} a_nx^n \text{ converges at } x = x_1 \implies \sum_{n=0}^{\infty} a_nx^n \text{ is A.C. for every } x \text{ with } |x| < |x_1|. \tag{2}$$

Then let $S = \{|x| : \sum_{n=0}^{\infty} a_nx^n \text{ converges}\}$. Notice that trivially $0 \in S$ always. If $S = \{0\}$ we are in case (i) and there is nothing further to prove. If S is not bounded above then (2) evidently implies that the series converges absolutely for all $x \in \mathbb{R}$, and we are in case (ii), so again there is nothing further to prove in this case.

The final possibility is that S contains at least two elements and is bounded above. Let $\rho = \sup S$; then $\rho > 0$, because S contains at least two elements. If $|x| > \rho$, then $\sum_{n=0}^{\infty} a_nx^n$ diverges (because otherwise we would have $|x| \in S$, and then by (2) that $y \in S$ for every y with $|y| < |x|$, so that $\rho \geq |x|$, a contradiction).

If on the other hand $|x| < \rho$, then there must be y such that $\sum a_n y^n$ converges and $|x| < |y|$ (otherwise $|x|$ would be a smaller upper bound for S than ρ). But then $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent by (2). Thus we have shown, in case S is non-empty and bounded above, the alternative (iii) holds with $\rho = \sup S$. This completes the proof.

Problems

4.1 Suppose the radius of convergence of $\sum_{n=0}^{\infty} a_n$ is 1, and the radius of convergence of $\sum_{n=0}^{\infty} b_n x^n$ is 2. Prove that the radius of convergence of $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ is 1.

4.2 If $\exists$ constant $c > 0$ such that $c^{-1} \leq |a_n| \leq c \ \forall n$, what can you say about the radius of convergences of $\sum_{n=0}^{\infty} a_n x^n$.

5. CONTINUITY OF REAL FUNCTIONS:

Here we shall mainly be interested in real-valued functions on some closed interval $[a, b] \subset \mathbb{R}$, i.e. functions $f : [a, b] \to \mathbb{R}$.

First recall the definition of continuity of such a function.

Definition 1: $f : [a, b] \to \mathbb{R}$ is said to be continuous at the point $c \in [a, b]$, if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$(*) \qquad |f(x) - f(c)| < \epsilon \text{ whenever } x \in [a, b] \text{ and } |x - c| < \delta.$$

Definition 2: We say $f : [a, b] \to \mathbb{R}$ is continuous if f is continuous at <u>each</u> point of $[a, b]$.

There is of course an algebra of real-valued continuous functions: the sum or product of continuous functions is also continuous. The quotient of continuous functions is also coninuous at points where the denominator is non-zero. As an exercise in the use of the definition of continuity, give rigorous proofs of these facts.

There are also other important properties given in the text with which you may be familiar, but we will cover these in lecture in the more general context of metric spaces, so we do not discuss them here.

Problems

5.1 Prove carefully (using the definition of continuity above) that the function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} +1 \text{ if } 0 < x \le 1 \\ 0 \text{ if } -1 \le x \le 0 \end{cases}$$

is not continuous at 0.

5.2 Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $|f| : [a, b] \to \mathbb{R}$ be defined by $|f|(x) = |f(x)|$. Prove that $|f|$ is continuous.

5.3 If $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0, 1] \text{ is a rational number} \\ 0 \text{ if } x \in [0, 1] \text{ is not rational} \end{cases},$$

prove that f is continuous at <u>no</u> point of $[0, 1]$.

Hint: Every open interval $(a, b) \subset \mathbb{R}$ ($a < b$ given) contains both rational and irrational numbers (see problem 1.8 above).

5.4 Suppose $f : (0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 \text{ if } x \in (0, 1] \text{ is not rational} \\ 1/q \text{ if } x \in (0, 1] \text{ is written } p/q \text{ where} \\ \qquad\qquad p, q \in \mathbb{N} \text{ have no common factor}. \end{cases}$$

Prove f is continuous at each irrational point of $(0, 1]$, and not continuous at each rational point of $(0, 1]$. (Hint: First note that for any given $\epsilon > 0$ there are at most finitely many positive integers q with $1/q \ge \epsilon$.)

6. COUNTABLE AND UNCOUNTABLE SETS

An infinite set (i.e., a set which is not finite) is said to countably infinite if its elements can be put into one to one correspondence with the set $\mathbb{N}$ of all positive integers.

That is, a set A is countably infinite if there is a map $f : A \to \mathbb{N}$ which is 1:1 and onto (i.e., a bijection). Notice that this is the same as saying that there is a sequence $a_1, a_2, \ldots$ of elements of A such that each element of A appears once and only once (just take $a_j = f^{-1}(j)$, $j = 1, 2, \ldots$).

For example, note that the set of all integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is countably infinite, because we can set up a bijection of $\mathbb{Z}$ onto $\mathbb{N}$ as follows:

$$f(n) = \begin{cases} 2n+1 & \text{if } n \text{ is non-negative} \\ -2n & \text{if } n \text{ is negative} \end{cases}$$

A set is said to be countable if either it is finite or if it countably infinite. (Note that "finite" includes the possibility that the set in question is empty.)

The following is a very useful criterion for countability:

Lemma 1. *A non-empty set A is countable $\iff$ there is a sequence $a_1, a_2, \ldots$ of elements of A such that each each element of A appears at least once.*

Proof: The direction "$\Rightarrow$" is clearly true by definition of countable; in case A is countable infinite there is a bijection g of A onto $\mathbb{N}$ and we can construct a suitable sequence $a_1, a_2, \ldots$ by defining $a_j = g^{-1}(j)$, $j = 1, 2, \ldots$, and in case A is a finite set $\{a_1, \ldots, a_N\}$ we can use the sequence $a_1, \ldots, a_N, a_1, \ldots, a_N, \ldots$.

Thus to complete the proof of Lemma 1 we just need to prove "$\Leftarrow$." So assume that there is a sequence $a_1, a_2, \ldots$ of elements of A such that each element of A appears at least once. If A is finite then it is countable by definition, so there is nothing to prove. If A is infinite we construct a mapping $f : \mathbb{N} \to A$ inductively by defining $f(1) = a_1$, and, assuming that $n \geq 2$ and that $f(1), \ldots, f(n-1)$ already defined,

$$(1) \qquad f(n) = a_k, \quad k = \min\{j : a_j \in A \setminus \{f(1), \ldots, f(n-1)\}\}.$$

Notice that f is well defined because $\{j : a_j \in A \setminus \{f(1), \ldots, f(n-1)\}\}$ is non-empty (by virtue of the assumption that A is infinite) and also because any non-empty set of positive integers does have a minimum. (This is the "well-ordering" property of the positive integers—see problem 1.1 above.)

We also note that f defined by (1) is both one to one and onto. (See Exercise 6.1 below.) Thus f is a bijection of $\mathbb{N}$ onto A, so A is countably infinite by definition.

Lemma 2. $\mathbb{N} \times \mathbb{N}$ *is countable.*

Proof: The map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(i,j) = \frac{1}{2}(i+j-1)(i+j-2) + i$ is a bijection. (See Exercise 6.2 below.)

There are some extremely useful corollaries of the above lemmas:

Corollary 1. *Any subset of a countable set is countable.*

Proof: This is an easy consequence of the above Lemma 1, and is left as an exercise.

Corollary 2. *If A, B are countable, then the cartesian product $A \times B$ is also countable.*

Proof: If either A or B is the empty set, then $A \times B$ is also the empty set, which is finite (hence countable) by definition, so we can assume that both A and B are non-empty.

By Lemma 1, there are sequences $a_1, a_2, \ldots,$ and $b_1, b_2, \ldots,$ in which every element of A and every element of B appear at least once. Let $g : \mathbb{N} \times \mathbb{N} \to A \times B$ be defined by $f(i,j) = (a_i, b_j)$, and let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection. (f exists by Lemma 2.) Then $h = g \circ f$ is a map of $\mathbb{N}$ onto $A \times B$; that is, $h(1), h(2), \ldots$ is a sequence in which every element of $A \times B$ appears at least once. By Lemma 1, this shows that $A \times B$ is countable.

Corollary 3. *The set $\mathbb{Q}$ of all rational numbers is a countable set.*

Proof: Let $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ be a bijection (f exists by Corollary 2), and define a map $g : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ by $g(i,j) = i/j$. Evidently g is onto (but not one to one), and hence $h = g \circ f$ is a map of $\mathbb{N}$ onto $\mathbb{Q}$. Thus $h(1), h(2), \ldots$ is a sequence in which every rational appears at least once. This proves the required result by virtue of Lemma 1.

Corollary 4. *If $A_1, A_2, \ldots$ is a sequence of sets, each of which is countable, then the union $\cup_{i=1}^{\infty} A_i$ is also countable.*

Note: Of course this means that the union of a finite collection $A_1, \ldots, A_N$ of countable sets is countable, because $\cup_{i=1}^{N} A_i = \cup_{i=1}^{\infty} A_i$, where we define $A_i = A_N$ for all $i \geq N$.

Proof of Corollary 4: We can assume $\cup_{i=1}^{\infty} A_i \neq \emptyset$, otherwise there is nothing to be proved. Let x be any element of $\cup_{i=1}^{\infty} A_i$. By Lemma 1, for each $i = 1, 2, \ldots$ such that $A_i \neq \emptyset$ there is a sequence $a_1^{(i)}, a_2^{(i)}, \ldots$ in which every element of A_i appears at least once. If $A_i = \emptyset$, let $a_j^{(i)} = x$ for each $j = 1, 2, \ldots$. By Lemma 2 there is a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Let $g : \mathbb{N} \times \mathbb{N} \to \cup_{i=1}^{\infty} A_i$ be defined by $g(i,j) = a_j^{(i)}$. Then $h = g \circ f$ is a map from $\mathbb{N}$ onto $\cup_{i=1}^{\infty} A_i$, and hence every element of $\cup_{i=1}^{\infty} A_i$ appears at least once in the sequence $h(1), h(2), \ldots$. The required result is thus established by Lemma 1.

Finally we want to give the proof of the fact that $\mathbb{R}$ is not countable; in fact we prove the more general result that no closed interval $[a,b]$ (where $a < b$) is countable.

Theorem. *Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed interval $[a,b]$ is not countable.*

Proof: Suppose on the contrary that $[a,b]$ is countable; it is evidently not finite, so this would mean that it is countably infinite. Thus we can write down a sequence $a_1, a_2, \ldots$ in which each element appears once and only once.

We inductively define a sequence $I_1, I_2, \ldots$ of subintervals of $[a,b]$ as follows:

We divide $[a,b]$ into 3 closed subintervals of equal length (each of length $\frac{1}{3}(b-a)$), and select one (call it I_1) which does <u>not</u> contain a_1. Assuming $n \geq 2$ and that I_{n-1} is already chosen, we divide I_{n-1} into 3 subintervals of equal length and select one (call it I_n) with the property $a_n \notin I_n$. Let $I_n = [c_n, d_n]$. Since by construction $I_n \subset I_{n-1}$ and length I_n $= \frac{1}{3}$ length I_{n-1} for each $n \geq 2$, we then have

$$a \leq c_1 \leq \cdots \leq c_{n-1} \leq c_n < d_n \leq d_{n-1} \cdots d_1 \leq b$$

and $d_n - c_n = 3^{-n}(b-a)$. Thus $\{c_n\}$, $\{d_n\}$ are monotone sequences with the same limit $y \in [a,b]$, and $y \in [c_n, d_n]$ for all n. But then $y \neq a_n$ for each $n \geq 1$, by virtue of the fact that $a_n \notin [c_n, d_n]$ by construction.

This contradicts the fact that $\{a_n\}$ is supposed to include all reals in $[a, b]$, because we have constructed a real $y \in [a, b]$ with $y \neq a_n$ for all $n \geq 1$.

Problems

6.1 Prove that the mapping f defined in (1) of the proof of Lemma 1 is indeed a bijection as claimed. Hint: To show that there is n such that $f(n) = a_k$, suppose there is no such n and show that this gives a contradiction.

6.2 (i) Prove that for each $n \in \mathbb{N}$ there are unique $m, k \in \mathbb{N}$ such that $k \leq m - 1$ and $n = \frac{1}{2}(m-1)(m-2) + m - k$.

Hint: Note first that $\frac{1}{2}(q-1)(q-2) \geq \frac{1}{2}(m-1)(m-2) + m - 1$ if $q \geq m + 1$.

(ii) Prove that the map $(i, j) \to \frac{1}{2}(i + j - 1)(i + j - 2) + i$ is a one-to-one map of $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$.

6.3 Prove that every infinite set has a countably infinite subset.

Hint: Describe an inductive procedure for finding a sequence $a_1, a_2, \ldots$ of distinct elements of the set.

6.4 Prove that the plane $\mathbb{R}^2 \equiv \{(x, y) : x \in \mathbb{R},\, y \in \mathbb{R}\}$ cannot be expressed as the union of a countable family of straight lines.

Hint: We already know that the x-axis is not countable.