Math 171 Final Examination

December 10, 2007

Name____________________________________

Signature_________________________________

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Total	

Directions:

1. This is closed book/notes exam (only your two 8 1/2 by 11 inch information sheets allowed).

2. Your signature above indicates that you accept the University Honor Code.

3. Write your solutions on the exam sheet; you may use the back side of a page if you run out of space. Throughout the exam you should give complete and clear proofs of your statements, justifying your steps. If you are using a particular theorem, be sure to state clearly what you are using. If you have a question about what you may assume without proof, please be sure to ask.

4. This test is 3 hours and has 7 problems worth a total of 80 pts.

5. Good luck!

Problem 1. Let A be a subset of a metric space M, and let $\partial A = \overline{A} \cap \overline{M \setminus A}$ denote the boundary of A (the book uses the notation $bd(A)$ instead of ∂A).

(a) (5 pts) Prove that A is open if and only if $A \cap \partial A = \emptyset$

(b) (5 pts) Prove that $x \in \partial A \iff$ there is a sequence of points from A converging to x and there is a sequence of points from $M \setminus A$ converging to x.

Problem 2. In this problem assume that M and N are metric spaces and that $f : M \to N$ is a *uniformly continuous* map.

(a) (5 pts) Prove that $f(M)$ is totally bounded if M is totally bounded.

(b) (5 pts) Show that the image under f of a Cauchy sequence from M is Cauchy in N. Give an example of a bounded continuous function f on $(0, 1)$ and a Cauchy sequence x_n in $(0, 1)$ such that $f(x_n)$ is not Cauchy.

Problem 3. A function g on $[0,1]$ is increasing if $g(x) \le g(y)$ for $x \le y$ with $x, y \in [0,1]$.

(a) (5 pts) Let f be continuous on $[0,1]$, differentiable on $(0,1)$, and assume that f' is uniformly continuous on $(0,1)$. Prove that f can be written in the form $f = g - h$ where g and h are continuous increasing functions on $[0,1]$.

(b) (5 pts) Show that the function $f(x) = x\sin(1/x)$ for $x > 0$ and $f(0) = 0$ cannot be written in the form $f = g - h$ where g and h are continuous increasing functions on $[0,1]$. (Hint: For an increasing continuous function g we have $\sum_{n=2}^{\infty} |g(x_n) - g(x_{n-1})| = g(x_1) - g(0)$ for any decreasing sequence x_n with $x_n \to 0$.)

Problem 4. A sequence of functions f_n on $[0, 1]$ is said to be increasing if $f_{n+1}(x) \geq f_n(x)$ for $x \in [0, 1]$, $n \geq 1$.

(a) (8 pts) Assume that f_n is an increasing sequence of continuous functions on $[0, 1]$ which converges pointwise to a continuous function f. Show that $\int_0^1 f_n \to \int_0^1 f$. You may *not* use the convergence theorems for the Lebesgue integral. (Hint: Prove uniform convergence.)

(b) (7 pts) Construct an increasing sequence f_n of continuous functions on $[0, 1]$ with $0 \leq f_n(x) \leq 1$ for all n and $x \in [0, 1]$ such that the limit function f is not Riemann integrable. Justify your answer. (You may use Lebesgue's characterization of Riemann integrable functions.)

Problem 5. (a) (5 pts) Let f_n be a sequence of C^1 functions on $(0,1)$. Assume that the sequence $f_n(1/2)$ converges and that f_n' converges uniformly to a function g on $(0,1)$. Show that f_n converges pointwise to a C^1 function f on $(0,1)$, and that $f' = g$. (You may use theorems from class and the book provided you make it clear what you are using.)

(b) (5 pts) Show that the series $\sum_{n=1}^{\infty}(-1)^n[\pi/2 - \tan^{-1}(nx)]$ converges for $x \in (1,2)$, and that the derivative of the sum is the sum of the derivatives on $(1,2)$. (You may use without proof the following information about $\tan^{-1}$: $\tan^{-1}(x)$ is defined and increasing for all x, takes values in $(-\pi/2, \pi/2)$, $\lim_{x\to\infty}\tan^{-1}(x) = \pi/2$, and $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.)

Problem 6. (a) (5 pts) Let $A_n \subseteq \mathbb{R}$ be a set of measure zero for $n \geq 1$. Prove directly from the definition that $\cup_{n=1}^{\infty} A_n$ has measure zero.

(b) (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz map. Show that if A has measure zero, then $f(A)$ has measure zero.

(c) (5 pts) Recall that the Cantor set C can be written as the intersection of subsets C_n of $[0, 1]$ where $C_n = \cup\{[k3^{-n}, (k+1)3^{-n}] : 0 \leq k \leq 3^n - 1,\ k \in \hat{Z}\}$ where $\hat{Z}$ denotes the set of integers whose base 3 expansion contains only the digits 0 and 2. Show that C has measure zero.

Problem 7. In this problem we consider the initial value problem $\frac{dx}{dt} = f(t, x)$ with $x(0) = x_0$. We assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and that there is a number $C > 0$ so that $0 \leq f(t, x) \leq C|x|$ for all (t, x). We also assume that for any $b > 0$ there exists a K depending on b such that $|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|$ for $(t, x) \in (-b, b) \times (-b, b)$.

(a) (5 pts) Prove that the initial value problem has a unique solution on the interval $[0, \infty)$. (Hint: Establish a bound on a solution defined on $[0, T)$ by a constant depending only on x_0, C, T, and use the local existence theorem to extend the solution to a larger interval.)

(b) (5 pts) For any $B > 0$, let $\mathcal{S} \subseteq C([0, 1])$ be the set of solutions (restricted to the interval $[0, 1]$) of the initial value problem with $x_0 \in [-B, B]$. Prove that $\mathcal{S}$ is a compact subset of $C([0, 1])$ where we take the L^∞ norm on $C([0, 1])$.