Math 171, Autumn 2007
Fall 2007 Final Exam Solutions

1) a) Assume that A is open. This implies that $M \setminus A$ is closed and therefore $A \cap \overline{M \setminus A} = A \cap (M \setminus A) = \emptyset$. Since $\partial A \subseteq \overline{M \setminus A}$ it follows that $A \cap \partial A = \emptyset$.

Assume that $A \cap \partial A = \emptyset$. Since $A \subseteq \overline{A}$, it follows that $A \cap \overline{M \setminus A} = \emptyset$, and thus $\overline{M \setminus A} = M \setminus A$ so $M \setminus A$ is closed and A is open.

b) Assume that $x \in \partial A$. Since $x \in \overline{A}$, there is a sequence from A converging to x. Since $x \in \overline{M \setminus A}$ there is a sequence from $M \setminus A$ converging to x.

Assume that there is a sequence from A converging to x and there is a sequence from $M \setminus A$ converging to x. Then it follows that $x \in \overline{A}$ and $x \in \overline{M \setminus A}$ so $x \in \partial A$.

2) a) If $\epsilon > 0$ is chosen, then by uniform continuity there exists $\delta > 0$ such that $\rho(f(x), f(\hat{x})) < \epsilon$ whenever $x, \hat{x} \in A$ with $d(x, \hat{x}) < \delta$. Because A is totally bounded we can find points $x_1, \ldots, x_N \in A$ so that $A \subseteq \cup_{n=1}^{N} D(x_n, \delta)$. We now show that $f(A) \subseteq \cup_{n=1}^{N} D(f(x_n), \epsilon)$. To see this, let $y \in f(A)$, then there exists $x \in A$ such that $f(x) = y$. It follows that $x \in D(x_n, \delta)$ and thus $d(x, x_n) < \delta$. Therefore $\rho(f(x), f(x_n)) < \epsilon$ and thus $y = f(x) \in D(f(x_n), \epsilon)$. Thus we have shown that $f(A) \subseteq \cup_{n=1}^{N} D(f(x_n), \epsilon)$ as required.

b) Let x_n be a Cauchy sequence from M. Given $\epsilon > 0$, there exists $\delta > 0$ so that $\rho(f(x), f(\hat{x})) < \epsilon$ for $x, \hat{x} \in M$ with $d(x, \hat{x}) < \delta$. Since x_n is Cauchy there exists N so that $d(x_n, x_m) < \delta$ for $n, m \geq N$. It follows that $\rho(f(x_n), f(x_m)) < \epsilon$ for $n, m \geq N$. Therefore the sequence $f(x_n)$ is Cauchy in N.

Let $f(x) = \cos(1/x)$ for $x \in (0, 1)$ and let $x_n = \frac{1}{n\pi}$. We then have $f(x_n) = \cos(n\pi) = (-1)^n$, so $f(x_n)$ is not a Cauchy sequence.

3) a) By the fundamental theorem of calculus we can write

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(s)ds$$

for $x, x_0 \in (0, 1)$. Since f' is uniformly continuous on $(0, 1)$, it extends continuously to the endpoints, and we can define $f'(0)$ and $f'(1)$ so that f' is continuous on $[0, 1]$. In particular f' is Riemann integrable on $[0, 1]$, so we can let $x_0 \to 0$ to obtain

$$f(x) = f(0) + \int_{0}^{x} f'(s)ds$$

for $x \in [0,1]$. We now let $f'_+ = \max\{f', 0\}$ and $f'_- = \max\{-f', 0\}$, and we have $f' = f'_- - f'_-$. Both f'_+ and f'_- are Riemann integrable since they are continuous in $[0,1]$. Therefore we have $f = g - h$ on $[0,1]$ where

$$g(x) = f(0) + \int_0^x f'_+(s)ds,\ h(x) = \int_0^x f'_-(s)ds,$$

and both g and h are continuous increasing functions.

b) Let $x_n = \frac{1}{\pi/2+n\pi}$, and observe that $|f(x_n) - f(x_{n-1}| = x_n + x_{n-1}$ since $\sin(1/x_n) = 1$ for n even and $\sin(1/x_n) = -1$ for n odd. Note that this series is divergent. Now if it were true that $f = g - h$ where g and h are continuous increasing functions on $[0,1]$, then we would have $|f(x_n) - f(x_{n-1})| \leq |g(x_n) - g(x_{n-1})| + |h(x_n) - h(x_{n-1})|$, and as observed in the hint, the series would be convergent. This contradiction shows that f cannot be so represented.

4) a) We show that $f_n \to f$ uniformly. To see this, let $g_n = f - f_n$ and observe that g_n is a decreasing sequence of continuous functions converging pointwise to 0. We show that g_n converges uniformly to 0. Let $\epsilon > 0$, and observe that for all $x \in [0,1]$ there exists n_x such that $g_{n_x}(x) < \epsilon/2$. Since g_{n_x} is continuous at x, there exists $\delta_x > 0$ so that $g_{n_x}(y) < \epsilon$ for $|y - x| < \delta_x$. By compactness, choose $x_1, \ldots, x_K \in [0,1]$ so that $[0,1] \subseteq \cup_{n=1}^K (x_n - \delta_{x_n}, x_n + \delta_{x_n})$. Let $N = \max\{n(1), \ldots, n(K)\}$ where $n(k) = n_{x_k}$, and observe that for any $x \in [0,1]$ we have $|x - x_k| < \delta_{x_k}$ for some k. Therefore $g_{n(k)}(x) < \epsilon$. Since $n(k) \leq N$ and the sequence g_n is decreasing, it follows that $g_N(x) < \epsilon$ and therefore $g_n(x) < \epsilon$ for $n \geq N$ and for all $x \in [0,1]$. Therefore g_n converges uniformly to 0, so f_n converges uniformly to f. By our theorem on interchange of limits with integrals we can now conclude that $\int_0^1 f = \lim \int_0^1 f_n$.

b) Enumerate the rational numbers in $[0,1]$ as a sequence r_n, and for $\epsilon > 0$ choose open intervals I_n centered at r_n of length $\epsilon 2^{-n}$. Let g_n be the piecewise linear function which is 0 outside I_n, positive inside I_n, and equal to 1 at r_n. We then let $f_n = \max\{g_1, \ldots, g_n\}$, and we see that f_n is an increasing sequence of continuous functions with $0 \leq f_n \leq 1$. Let f be the pointwise limit of f_n, and we claim that f is not Riemann integrable. To see this, let $K = [0,1] \setminus \cup_{n=1}^\infty I_n$, and observe that for any $x \in K$, the function f is not continuous at x since $f(x) = 0$ and any open interval about x contains points r_n at which the value of f is 1.

We claim now that K does not have measure zero and therefore f is not Riemann integrable by Lebesgue's theorem. To see this observe that if K is covered by a collection of open intervals J_k, then the J_k together with the I_n form an open covering of $[0,1]$ which then has a finite subcovering. Now the total length of the intervals in the finite subcovering is at least 1 since the intervals cover $[0,1]$, but the total length of the I_n is ϵ. It follows that the total length of the J_k is at least $1-\epsilon$, and thus K does not have measure zero if $\epsilon < 1$.

5) a) By the fundamental theorem of calculus we can write

$$f_n(x) = f_n(1/2) + \int_{1/2}^{x} f_n'(s)ds.$$

For any given $x \in (0,1)$ it follows from the uniform convergence of the sequence f_n' and the convergence of the sequence $f_n(1/2)$ that $f_n(x)$ converges to a limit $f(x)$, and

$$f(x) = f(1/2) + \int_{1/2}^{x} g(s)ds.$$

From this it follows again by the fundamental theorem of calculus that f is differentiable on $(0,1)$ and $f' = g$.

b) Let $f_n = (-1)^n[\pi/2 - \tan^{-1}(nx)]$, and observe that for each $x \in (1,2)$ the series converges since the terms alternate in sign and are decreasing to zero in absolute value. Therefore the series converges pointwise in $(1,2)$. Now we have $f_n'(x) = (-1)^{n+1}\frac{n}{1+n^2x^2}$, and this series also converges for the same reason. From the theory of alternating series we also know that the sum of the series $g(x)$ satisfies $|g(x) - g_n(x)| \leq |f_{n+1}'(x)|$ where g_n denotes the partial sum $g_n(x) = \sum_{k=1}^{n} f_n'(x)$. Since the $|f_n'(x)| < 1/n$ for all $x \in (1,2)$ it follows that the series of derivatives converges uniformly to a function g on $(1,2)$, and therefore it follows that the series can be differentiated term by term.

6) a) Let $\epsilon > 0$ be given. Since A_n has measure zero, there is a countable collection $\mathcal{I}_n$ of open intervals so that $A_n \subseteq \cup_{I\in\mathcal{I}_n} I$ and $\sum_{I\in\mathcal{I}_n} |I| < \epsilon 2^{-n}$. Let $\mathcal{I} = \cup_{n=1}^{\infty}\mathcal{I}_n$. Since a countable union of countable sets is countable we see that $\mathcal{I}$ is a countable collection of open intervals. We have $\sum_{I\in\mathcal{I}} |I| = \sum_{n=1}^{\infty}\sum_{I\in\mathcal{I}_n} |I| < \sum_{n=1}^{\infty} \epsilon 2^{-n} = \epsilon$. Moreover we have $\cup_{n=1}^{\infty} A_n \subseteq \cup_{I\in\mathcal{I}} I$, so it follows that $\cup_{n=1}^{\infty} A_n$ has measure zero.

b) Assume that $|f(x) - f(y)| \leq K|x-y|$, and observe that if I is any finite open interval, then $f(I)$ is an interval of length at most $K|I|$, and therefore it is contained in an open interval J_I of length $2K|I|$. Let $\epsilon > 0$ and let $\mathcal{I}$ be a collection of open intervals which cover A and with $\sum_{I\in\mathcal{I}} |I| < \frac{\epsilon}{2K}$. For each $I \in \mathcal{I}$, let J_I be an open interval which contains $f(I)$ with $|J_I| = 2K|I|$. We then have $f(A) \subseteq \cup_{J\in\mathcal{I}} J_I$, and $\sum_{I\in\mathcal{I}} |J_I| = 2K \sum_{I\in\mathcal{I}} |I| < \epsilon$. Therefore $f(A)$ has measure zero.

c) Since C_n is a union of 2^n intervals of length 3^{-n}, for any $\lambda > 1$ we can cover C_n by 2^n *open* intervals of length at most $\lambda^n 3^{-n}$. This collection of open intervals then covers $C \subseteq C_n$, and the total length of the intervals is at most $(2\lambda/3)^n$. If λ is chosen smaller than $3/2$ we will have $(2\lambda/3)^n < \epsilon$ for n sufficiently large. Therefore C has measure zero.

7) a) Let $x(t)$ be the solution of the initial value problem with $x(0) = x_0$ and assume that $x(t)$ is defined on $[0, T)$. Since $f \geq 0$ we see that $x(t)$ is an increasing function of t so we have the lower bound $x_0 \leq x(t)$ for all $t \in [0, T)$. If $x(t) \leq 1$ for all $t \in [0, T)$ then we have the desired bound. Assume that there exists $t_0 \in [0, T)$ so that $x(t_0) = 1$ and thus $x(t) \geq 1$ for $t \in [t_0, T)$. Since $x'(t) = f(t, x(t)) \leq Cx(t)$ for $t \in [t_0, T)$, we have $(\log x)' \leq C$ and therefore integrating over $[t_0, t]$ we find $\log x(t) \leq C(t - t_0)$ for $t \in [t_0, T)$. Therefore $x(t) \leq e^{C(t-t_0)} \leq e^{CT}$ for all $t \in [t_0, T)$. We have shown $x_0 \leq x(t) \leq e^{CT}$ for all $t \in [0, T)$.

We now let S be the subset of $[0, \infty)$ consisting of those T such that the initial value problem has a unique solution on $[0, T)$ which extends continuously to $[0, T]$. By the local existence and uniqueness theorem we see that $[0, \delta) \subseteq S$ so S is not empty. We show that S is both open and closed, and therefore by connectedness of $[0, \infty)$ we must have $S = [0, \infty)$. First note that S is open by the Extension Principle. To see that S is closed, suppose that $T_n \in S$ and $T_n \to T$. If any $T_n \geq T$, then $T \in S$, so we may assume that $T_n < T$ for all n. It then follows that the initial value problem has a solution on $[0, T)$. This solution is unique by the global uniqueness theorem. Since the solution $x(t)$ is increasing and bounded above (by the estimate in the previous paragraph), it follows $x(t)$ extends continuously to $[0, T]$, and therefore $T \in S$. This shows that S is closed, and therefore there exists a solution of the initial value problem defined on $[0, \infty)$. The solution is unique by the global uniqueness theorem.

b) From the estimate of part a we see that the collection $\mathcal{S}$ is uniformly bounded. Since $x' = f(t, x(t))$ for each function $x(t)$ in $\mathcal{S}$ it follows that $|x'(t)|$ is uniformly bounded for $t \in [0, 1]$ and therefore $x(t)$ is Lipschitz with a fixed Lipschitz constant. Therefore $\mathcal{S}$ is equicontinuous. By the Arzela-Ascoli theorem we see that $\mathcal{S}$ has compact closure, so it remains to show that $\mathcal{S}$ is a closed subset of $C([0, 1])$. To see this, let $x_n(t)$ be a convergent sequence from $\mathcal{S}$ and let $x(t)$ be its limit. We then have by the fundamental theorem of calculus

$$x_n(t) = x_n(0) + \int_0^t f(s, x_n(s))ds.$$

Since x_n converges uniformly to x, we have

$$|f(s, x_n(s)) - f(s, x(s)| \leq K|x_n(s) - x(s)|$$

where K depends only on the bound on the functions in $\mathcal{S}$. It follows that the sequence $f(s, x_n(s))$ converges uniformly to $f(s, x(s))$ and so we can take the limit and get

$$x(t) = x(0) + \int_0^t f(s, x(s))ds.$$

Therefore we have $x'(t) = f(t, x(t))$ and $x(0) \in [-B, B]$. Thus $x \in \mathcal{S}$ and therefore $\mathcal{S}$ is closed.