1. Let $A \subseteq \mathbb{R}$ have Lebesgue measure zero. Prove that $B = A \times \mathbb{R}$ has Lebesgue measure zero.

2. Let $R \subseteq \mathbb{R}^n$ be a rectangle and let $f : R \to \mathbb{R}$ be a Lebesgue integrable function. Prove that if $g : R \to \mathbb{R}$ is another function and $f = g$ a.e., then g is also Lebesgue integrable, and $\int_R f = \int_R g$. [Hint: First consider the case $f \in \mathcal{L}_+(R)$.]

3. Let $\{q_n\}$ be a sequence such that $\mathbb{Q} \cap [0,1] = \{q_n | n \in \mathbb{N}\}$. For $t > 0$, define a subset $U_t \subseteq \mathbb{R}$ as

$$U_t = \cup_{n=1}^{\infty} (q_n - t2^{-n}, q_n + t2^{-n}).$$

(a) Prove that U_t is open (in $\mathbb{R}$).

(b) Prove that the closure of U_t contains $[0,1]$.

(c) Prove that if $t < 1/2$ then $[0,1] \setminus U_t$ does *not* have measure zero. [Hint: one of the problems from HW7 is useful.]

(d) Let $f : [0,1] \to \mathbb{R}$ be defined by $f = \chi_{[0,1] \cap U_{1/3}}$. Prove that $f \in \mathcal{L}_+([0,1])$.

(e) Let $g : [0,1] \to \mathbb{R}$ be defined by $g(x) = 1 - f(x)$. Prove that $g \in \mathcal{L}([0,1])$ and $g \geq 0$, but $g \notin \mathcal{L}_+([0,1])$.

4. Let $R \subseteq \mathbb{R}^n$ be a rectangle. As usual, we shall write $\mathcal{S}(R)$ for the set of step functions $R \to \mathbb{R}$.

(a) Let $\phi, \psi \in \mathcal{S}(R)$. Prove that $\max(\phi, \psi)$ is again a step function.

(b) Deduce that if $f, g \in \mathcal{L}_+(R)$, then $\max(f, g) \in \mathcal{L}_+(R)$.

5. In a previous problem (practice problems for midterm) you proved that the inclusion $i : \ell^1 \to \ell^\infty$ (i.e. the function which sends a sequence to itself, but now regarded as an element of ℓ^∞) is continuous. Let $M \subseteq \ell^\infty$ be the image of i, regarded as a metric space using the relative metric from ℓ^∞. Prove that $i^{-1} : M \to \ell^1$ is *not* continuous.

6. The goal of this problem is to prove the following theorem: Let $f_n : [-1,1] \to [-1,1]$ be a continuous function for each $n \in \mathbb{N}$ (not necessarily surjective). Then there exists a sequence of elements $a_n \in [-1,1]$ such that $f_n(a_{n+1}) = a_n$ for all n. Proceed in the following steps, where H^∞ denotes the Hilbert cube (everything you've previously proved about the Hilbert cube may be used without proof in this exercise).

(a) For each n, let $X_n \subseteq H^\infty$ be the subset consisting of elements $a = (a_1, a_2, \dots)$ such that $f_n(a_{n+1}) = a_n$. Prove that X_n is a closed subset of H^∞.

(b) Define subsets $Y_n \subseteq H^\infty$ by $Y_0 = H^\infty$ and inductively $Y_n = Y_{n-1} \cap X_n$. Prove that each $Y_n \subseteq H^\infty$ is non-empty.

(c) Prove that $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots$ and that all Y_n are closed subsets of H^∞.

(d) Use compactness of H^∞ to deduce that $Y = \cap_{n=1}^\infty Y_n$ is non-empty. (This proves the theorem.)

(Bonus question 1: Modify the assumptions to having a closed non-empty set $A_n \subseteq [-1, 1]$ for each n and continuous functions $f_n : A_{n+1} \to A_n$.)

(Bonus question 2: Would the theorem be true if we replaced $[-1, 1]$ by $(-1, 1)$?)

7. Let $f : [0, \infty) \to [0, \infty)$ be the function given by $f(x) = \sqrt{x}$. Prove that f is uniformly continuous.

8. Let M_1, M_2 and M_3 be metric spaces, and let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be uniformly continuous. Prove that $g \circ f$ is uniformly continuous.

9. Let $b = \{b_n\}_{n=1}^\infty$ be a sequence of positive real numbers and define a subset $X_b \subseteq \ell^\infty$ by

$$X_b = \{a = \{a_n\}_{n=1}^\infty \in \ell^\infty \mid |a_n| \le b_n \text{ for all } n\}.$$

We shall consider X_b a metric space using the relative metric from ℓ^∞.

(a) Prove that if X_b is compact, then $\lim_{n\to\infty} b_n = 0$.

(b) Prove that if $\lim_{n\to\infty} b_n = 0$, then X_b is compact.

(c) Prove that if $0 < c_n \le b_n$ and $\lim b_n = 0$ then the function $f : X_b \to X_c$ given by

$$f(\{a_n\}_{n=1}^\infty) = \{\frac{c_n}{b_n} a_n\}_{n=1}^\infty$$

is continuous.

(d) Prove that f is a bijection and that $f^{-1} : X_c \to X_b$ is continuous.

10. Let (M, d) be a compact metric space. Let d' be another metric on the same space, and assume that if a subset $U \subseteq M$ is open with respect to d', then it is also open with respect to d. Prove that d and d' are equivalent metrics.