MATH 171: MIDTERM
THURSDAY, MAY 5, 2011

This is a closed book, closed notes, no calculators/computers exam. As usual, $\mathbb{P}$ denotes the set of positive integers, and $\mathbb{R}^n$ comes with the Euclidean metric unless otherwise specified.

You may quote any theorem from the textbook or the lecture provided that you are not explicitly asked to prove it, and provided you state the theorem precisely and concisely (make sure to check the hypotheses in writing when you quote a theorem).

There are 5 problems. Solve all of them. Write your solutions to Problems 1 and 2 in blue book #1, and your solutions to Problems 3-5 in blue book #2 to facilitate grading. You may solve the problems in any order.

Problem 1. *(15 points)*

(i) State the definition of the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.
(ii) State the definition that a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is bounded.
(iii) Suppose that $\{a_n\}$ is a bounded sequence of real numbers. Let $b_n = a_n/n$. Show directly from the definition of the limit (i.e. without using limit theorems) that $\lim_{n\to\infty} b_n = 0$.

Problem 2. *(20 points)* Suppose (M, d) is a metric space.

(i) State the definition of a subset C of M being closed.
(ii) For $x \in M$ and $r > 0$ let $\bar{B}_r(x) = \{y \in M : \; d(y, x) \leq r\}$. Show that $\bar{B}_r(x)$ is closed. (It is called the closed ball of radius r around x.)

Problem 3. *(15 points)* Suppose that $A \neq \emptyset$ is a bounded subset of $\mathbb{R}$ and has the following property:

$$x, z \in A \text{ and } x < y < z \Rightarrow y \in A.$$

Let $a = \inf A$, $b = \sup A$. Show that

$$a < y < b \Rightarrow y \in A.$$

Conclude that A must be one of the following intervals: (a, b), $(a, b]$, $[a, b)$, $[a, b]$ (with $[a, b]$ being the only possibility if $a = b$).

Problem 4. *(25 points)* Recall that ℓ^2 is the vector space of square summable sequences with $\|\{a_n\}\|_{\ell^2} = \sqrt{\sum_{n=1}^{\infty} a_n^2}$, and ℓ^∞ is the vector space of bounded sequences with $\|\{a_n\}\|_{\ell^\infty} = \sup_{n\in\mathbb{P}} |a_n|$.

(i) Show that $\ell^2 \subset \ell^\infty$.
(ii) Show that if $a, a^{(k)} \in \ell^2$, $k \in \mathbb{P}$, and $\lim_{k\to\infty} a^{(k)} = a$ in ℓ^2 then $\lim_{k\to\infty} a^{(k)} = a$ in ℓ^∞.
(iii) Give an example of $a, a^{(k)} \in \ell^2$, $k \in \mathbb{P}$ such that $\lim_{k\to\infty} a^{(k)} = a$ in ℓ^∞ but $a^{(k)}$ does *not* converge to a in ℓ^2.

Problem 5. *(25 points)* If $(M_1, d_1), (M_2, d_2)$ are metric spaces, let $M = M_1 \times M_2$ be the metric space with $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$. We call d the *product metric* on M.

(i) Show that the product metric on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ is equivalent to the Euclidean metric on $\mathbb{R}^{2n}$, i.e. a set is open in the product metric if and only if it is open in the Euclidean metric on $\mathbb{R}^{2n}$.
(ii) Show that $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.
(iii) Show that if (X, ρ), (M_1, d_1) and (M_2, d_2) are metric spaces, $f_1 : X \to M_1$ and $f_2 : X \to M_2$ are continuous then $f : X \to M_1 \times M_2$ defined by $f(x) = (f_1(x), f_2(x))$ is continuous.
(iv) With X as in (iii), show that if $f_1 : X \to \mathbb{R}^n$ and $f_2 : X \to \mathbb{R}^n$ are continuous then $f_1 + f_2 : X \to \mathbb{R}^n$, given by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, is also continuous.