Math 171: Midterm Solutions

Spring 2013

1. (a) Check the axioms:
   (i) Since $C > 0$, $d'(x, y) = \min(d(x, y), C) = 0$ iff $d(x, y) = 0$ iff $x = y$.
   (ii) $d'(x, y) = \min(d(x, y), C) = \min(d(y, x), C) = d'(y, x)$.
   (iii) We need to show that $\min(d(x, z), C) \leq \min(d(x, y), C) + \min(d(y, z), C)$. If either $d(x, y) \geq C$ or $d(y, z) \geq C$, then LHS $\leq C \leq$ RHS. If both $d(x, y) < C$ and $d(y, z) < C$, then LHS $\leq d(x, z) \leq d(x, y) + d(y, z) =$ RHS.

   (b) For $\epsilon \leq C$, $\min(d(x, y), C) < \epsilon$ iff $d(x, y) < \epsilon$, so $B_\epsilon(x) = B'_\epsilon(x)$ (open ϵ-balls for d and d', respectively). If U is open w.r.t. d, then for all $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Replacing ϵ by $\min(\epsilon, C)$, we may assume that $\epsilon \leq C$, so $B'_\epsilon(x) \subset U$. Thus U is open w.r.t. d. The same argument gives the converse.

2. For $\{a_n\} \in \ell^1$, $\sum a_n$ converges absolutely, hence converges. Thus f is well defined. For $\{a_n\}, \{b_n\} \in \ell^1$,

$$|f(\{a_n\}) - f(\{b_n\})| = \left|\sum a_n - \sum b_n\right| = \left|\sum (a_n - b_n)\right| \leq \sum |a_n - b_n| = d(\{a_n\}, \{b_n\}),$$

where d is the ℓ^1-metric. This gives the (uniform) continuity of f with $\delta = \epsilon$.

[More carefully, using triangle inequality first for finite sums and then passing to the limit: For all $N \geq 1$,

$$\left|\sum_{n=1}^{N} a_n - \sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} (a_n - b_n)\right| \leq \sum_{n=1}^{N} |a_n - b_n| \leq \sum_{n=1}^{\infty} |a_n - b_n| = d(\{a_n\}, \{b_n\}).$$

Since $x \mapsto |x|$ is continuous and the limit of a difference is the difference of the limits, taking $N \to \infty$ yields $|f(\{a_n\}) - f(\{b_n\})| \leq d(\{a_n\}, \{b_n\})$.]

3. Let $A = \limsup a_n$, $B = \limsup b_n$. Let $\epsilon > 0$. By the definition of lim inf and lim sup,

$$0 < a_n < A + \epsilon, \quad 0 < b_n < B + \epsilon \tag{1}$$

for all large n. Then $a_n b_n < (A+\epsilon)(B+\epsilon)$ for all large n, so $\limsup a_n b_n \le (A+\epsilon)(B+\epsilon)$. Taking $\epsilon \to 0$, $\limsup a_n b_n \le AB = (\limsup a_n)(\limsup b_n)$.

[More carefully: By the definition of $\liminf$, there exists N_1 such that

$$n \ge N_1 \text{ implies } a_n > 0.$$

Similarly, there exist N_2, N_3, N_4 such that

$$\begin{aligned} n &\ge N_2 \text{ implies } b_n > 0 \\ n &\ge N_3 \text{ implies } a_n < A + \epsilon \\ n &\ge N_4 \text{ implies } b_n < B + \epsilon. \end{aligned}$$

Thus $n \ge \max(N_1, N_2, N_3, N_4)$ implies (1).]

4. Let d be the ℓ^2-metric. For $n \in \mathbb{N}$, let $a_n = (\frac{1}{n}, 0, 0, \ldots) \in \ell^2$. Let $A = \{a_n : n \in \mathbb{N}\}$. As $n \to \infty$, $d(a_n, (0, 0, \ldots)) = \frac{1}{n^2} \to 0$, so $a_n \to (0, 0, \ldots) \in \ell^2 \setminus A$. Thus A is not closed.

5. (a) $(0, \infty)$ is open in $\mathbb{R}$, so $f^{-1}((0, \infty)) = \{x \in M : f(x) > 0\}$ is open is M.

   Alternatively: Let $x_0 \in U := \{x \in M : f(x) > 0\}$. By the continuity f at x_0 with $\epsilon = f(x_0)$, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $|f(x) - f(x_0)| < f(x_0)$, so in particular $f(x) > 0$. Thus $B_\delta(x_0) \subset U$. Since x_0 was arbitrary, this shows U is open.

   (b) Define $f(x) = \inf_{a \in U^c} d(x, a)$. Since $d(x, a) \ge 0$ for all $x, a \in M$, $f(x) \ge 0$ for all $x \in M$. If $x \in U^c$, then takig $a = x$, $f(x) \le d(x, x) = 0$, so $f(x) = 0$. Now let $x \in U$. Since U is open, $B_\epsilon(x) \subset U$ for some $\epsilon > 0$. If $a \in U^c$, then $a \notin B_\epsilon(x)$, i.e. $d(x, a) \ge \epsilon$. Taking infimum, $f(x) \ge \epsilon > 0$. Thus $U = \{x \in M : f(x) > 0\}$.

   Fix $x, y \in M$. For any $a \in U^c$, $d(x, a) \le d(x, y) + d(y, a)$. Taking infimum over $a \in U^c$, $f(x) \le d(x, y) + f(y)$. By symmetry, $f(y) \le d(x, y) + f(x)$ as well, so

   $$|f(x) - f(y)| \le d(x, y) \quad \text{for all } x, y \in M,$$

   which gives the (uniform) continuity of f with $\delta = \epsilon$.