**Math 171 Midterm**

5/2, 9:30am – 10:45am

Open book, open notes. No computers or internet enabled devices allowed. *Give complete proofs.*

**1.** (10 points) Let $(M, d)$ be a metric space and $C$ a positive real number. Define a function $d' : M \times M \to \mathbb{R}$ as $d'(x, y) = \min(d(x, y), C)$.

(a) Prove that $(M, d')$ is a metric space.

(b) Prove that a subset $U \subset M$ is open with respect to $d$ if and only if it is open with respect to $d'$.

**2.** (10 points) Prove that the function $f : \ell^1 \to \mathbb{R}$ defined by

$$f(\{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} a_n$$

is well defined and continuous.

**3.** (10 points) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be bounded sequences of real numbers. Assume that $\liminf a_n > 0$ and $\liminf b_n > 0$. Prove that

$$\limsup_{n\to\infty}(a_n b_n) \leq \big(\limsup_{n\to\infty} a_n\big)\big(\limsup_{n\to\infty} b_n\big).$$

**4.** (10 point) Give an example (with proof) of a subset $A \subset \ell^2$ which is *not* closed.

**5.** (10 points) Let $M$ be a metric space.

(a) Let $f : M \to \mathbb{R}$ a continuous function. Prove that the set

$$\{x \in M | f(x) > 0\}$$

is an open subset of $M$.

(b) Let $U \subseteq M$ be an open set. Prove that there exists a continuous function $f : M \to \mathbb{R}$ with $f(x) \geq 0$ for all $x \in M$ such that $U = \{x \in M | f(x) > 0\}$.