Math 171 Midterm Examination

May 1, 2014

Solutions

Name________________________________

Signature________________________________

1	10
2	10
3	10
4	10
5	10
Total	50

Directions:

1. This is an open book/open notes exam, but you may not use the internet during the exam.
2. Your signature above indicates that you accept the University Honor Code.
3. Write your solutions on the exam sheet; you may use the back side of a page if you run out of space. Throughout the exam you should give complete and clear proofs of your statements, justifying your steps. If you are using a particular theorem, be sure to state clearly what you are using. If you have a question about what you may assume without proof, please be sure to ask.
4. You have 2 hours to complete this test. It has 5 problems worth a total of 50 pts.
5. Good luck!

Problem 1. Let $\{a_n\}$ be a sequence of real numbers.

(a) (5 pts) Assume that there is a number $\epsilon > 0$ such that $|a_n - a_m| > \epsilon$ for $n \neq m$. Show that for any number $R > 0$ the set $\{n : |a_n| \leq R\}$ is finite.

The condition implies that $\{a_n\}$ has no Cauchy subsequence. If for some $R>0$ the set $\{n : |a_n| \leq R\}$ were infinite then it would contain a convergent subsequence by Bolzano-Weierstrass. This subsequence would be Cauchy which is a contradiction. Therefore the set is finite for all $R>0$.

(b) (5 pts) Suppose for each positive integer n we have $|a_{n+1} - a_n| < 2^{-n}$. Show that the sequence converges.

We have for any n, $k \geq 1$

$$|a_{n+k} - a_n| \leq |a_{n+k} - a_{n+k-1}| + \cdots + |a_{n+1} - a_n|$$

$$< \sum_{i=1}^{k} 2^{-n-i+1} = 2^{1-n} \sum_{i=1}^{k} 2^{-i} < 2^{1-n}$$

Given $\varepsilon > 0$, let N be large enough so that $2^{1-N} < \varepsilon$.

for $n, m \geq N$ where w.l.o.g we assume $m = n+k > n$.

we have $|a_m - a_n| < 2^{1-n} < \varepsilon$.

$\therefore$ $\{a_n\}$ is Cauchy and hence convergent (Cauchy criterion)

Problem 2. Let (M, d) be the metric space with $M = [1, \infty)$ and $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$.

(a) (7 pts) Show that a subset $O \subseteq M$ is open if and only if O is an open subset of $[1, \infty)$ with its usual distance function. (You may assume that if $d(x, y) < (2x)^{-1}$, then $|x - y| < 2x^2 d(x, y)$.)

($\Rightarrow$) Assume O is open in M. Let $x \in O$ & $\varepsilon > 0$ such that $B_\varepsilon^M(x) \subseteq O$. Since $d(x,y) \leq |x-y|$ it follows that $(x-\varepsilon, x+\varepsilon) \cap [1,\infty) \subseteq B_\varepsilon^M(x)$ and so O is open in $[1,\infty)$ with its usual distance.

($\Leftarrow$) Assume O is open in $[1,\infty)$ with its usual dist. Let $x \in O$ and $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \cap [1,\infty) \subseteq O$. Let $\delta = \min\{(2x)^{-1}, (2x^2)^{-1}\varepsilon\}$ and we see that if $d(x,y) < \delta$ then $|x-y| < 2x^2\delta \leq \varepsilon$, so $B_\delta^M(x) \subseteq O$ and O is open in M.

(b) (3 pts) Show that the set of positive integers is a closed bounded subset of M which is not compact.

$\mathbb{N}$ is closed in M since it is closed in $[0,\infty)$ with its usual distance (part (a)). Again by (a) each set $\{n\}$ for $n \in \mathbb{N}$ is open in $\mathbb{N}$, so we have $\mathcal{O} = \{\{n\} : n \in \mathbb{N}\}$ is an open cover with no finite subcover, so $\mathbb{N}$ is <u>not</u> compact. $\mathbb{N} \subseteq B_1^M(1) = [1,\infty)$, so $\mathbb{N}$ is bounded.

Problem 3. (a) (7 pts) Suppose M is a metric space, A is a subset of M, and f is a uniformly continuous real valued function defined on A. Show that there is a continuous function $\hat{f}$ on $\bar{A}$ such that $\hat{f}(x) = f(x)$ for $x \in A$.

Claim: If $\{x_n\}$ is Cauchy in A then $\{f(x_n)\}$ is Cauchy

Pf: $\varepsilon > 0 \Rightarrow \exists\ \delta$ s.t. $|f(x) - f(y)| < \varepsilon$ if $d(x,y) < \delta$

$\delta > 0 \Rightarrow \exists\ N$ s.t. $d(x_n, x_m) < \delta$ if $n, m \geq N$. Thus for $n, m \geq N$ we have $|f(x_n) - f(x_m)| < \varepsilon$ & $\{f(x_n)\}$ is Cauchy.

Given $x \in \bar{A}$, $\exists\ \{x_n\}$ in A with $\lim x_n = x$. Define $\hat{f}(x) = \lim f(x_n)$. If $x \in A$ we have $\hat{f}(x) = f(x)$ since f is continuous at x. If $x \notin A$ and $\{y_n\}$ is another sequence from A with $\lim y_n = x$, then we have $d(x_n, y_n) \to 0$ and thus by uniform continuity $|f(x_n) - f(y_n)| \to 0$ & hence $\lim f(y_n) = \lim f(x_n) = \hat{f}(x)$. Thus $\hat{f}$ is well defined. Since $|\hat{f}(x) - \hat{f}(y)| \leq \sup_{p,q \in A} |f(p) - f(q)|$ for $x, y \in \bar{A}$, we see that $\hat{f}$ is uniformly cont on $\bar{A}$.

(b) (3 pts) Give an example of a bounded continuous function on $(0,1)$ which is not uniformly continuous. Justify your answer.

$f(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $(0,1)$ but does not extend continuously to $[0,1]$ and so $f(x)$ is not uniformly continuous by (a).

Problem 4. (10 pts) Let M_1 and M_2 be metric spaces and let f be a map from M_1 to M_2. Show that f is continuous if and only if for all subsets A of M_1 we have $f(\bar{A}) \subseteq \overline{f(A)}$.

($\Rightarrow$) Assume f is continuous.

Let $y \in f(\bar{A})$, so $\exists\ x \in \bar{A}$ with $y = f(x)$.

Let $\{x_n\}$ in A with $\lim x_n = x$. Since f is continuous at x we have $\lim f(x_n) = f(x) = y$.

Thus y is a limit point of $f(A)$; that is, $y \in \overline{f(A)}$

($\Leftarrow$) Assume $f(\bar{A}) \subseteq \overline{f(A)}$ $\forall\ A \subseteq M_1$.

Let $x \in M_1$. Suppose f is not continuous at x.

$\exists\ \varepsilon_0 > 0$ so that $\forall\ \delta > 0$ $\exists\ y \in B_\delta(x)$ such that $f(y) \notin B_{\varepsilon_0}(f(x))$.

Take $\delta = 1/n$ and let y_n be such a point. Let A be the set consisting of the points $\{y_n : n = 1, 2, 3, \dots\}$.

Since $y_n \to x$ we see that $x \in \bar{A}$, but $f(y_n) \notin B_{\varepsilon_0}(f(x))$ $\forall n$

and so $f(x) \notin \overline{f(A)}$, a contradiction

$\therefore$ f is continuous at x for all $x \in M_1$.

Problem 5. Let M be a compact metric space. Suppose, for some positive integer n, we have nonempty closed subsets $F_1, \ldots, F_n$ such that $F_i \cap F_j = \phi$ for $i \neq j$, and such that $M = \cup_{i=1}^{n} F_i$. Let $n(M)$ be the maximum such integer n. We take $n(M) = \infty$ if n can be aribtrarily large.

(a) (7 pts) Suppose that for all $x \in M$ there exists $\epsilon > 0$ such that $B_\epsilon(x)$ is connected. Show that $n(M)$ is finite.

Since $F_i = \left(\bigcup_{j \neq i} F_j\right)'$ we see that each F_i is open.

Since $B_\varepsilon(x)$ is connected we must have $B_\varepsilon(x) \subseteq F_i$ for some i

Since M is compact we can cover M by a finite number of such balls $B_{\varepsilon_1}(x_1), \ldots, B_{\varepsilon_k}(x_k)$

Now each ball $B_{\varepsilon_i}(x_i)$ is contained in one and only one F_j, so it follows that $n \leq k$.

$\therefore\ n(M) \leq k < \infty$.

(b) (3 pts) Give an example of a compact metric space M with $n(M) = \infty$.

$M = \{0, 1, 1/2, 1/3, 1/4, \ldots\} = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subseteq \mathbb{R}$.

M is closed and bounded hence compact.

For any integer n we can take $F_i = \{1/i\}$ for $i = 1, \ldots, n$

and $F_{n+1} = M \setminus \bigcup_{i=1}^{n} F_i$. All sets are closed and pairwise disjoint, so $n(M) = \infty$.