Math 171 Writing in the Major (WIM) Assignment

1. Overview

The topic of your writing assignment is **completions of metric spaces**. A metric space M is *complete* if every Cauchy sequence in M is convergent (in M). We covered the definition of a complete metric space in class on Friday, May 6, and by Monday May 9 we will discuss many (but not all!) important examples of complete and non-complete metric spaces.

Although not every given metric space M is complete, one can try to repair this gap by "filling in the holes," or looking for a metric space $\widehat{M}$ containing M which is complete, and "as small as possible." Concretely, a *completion* of a metric space $M := (M, d)$ is a complete metric space $\widehat{M} := (\widehat{M}, \widehat{d})$ equipped with an *isometric injection* $i : M \to \widehat{M}$ which is *dense*, meaning that $\widehat{d}(i(x), i(y) = d(x, y)$, i is injective, and $\overline{i(M)} = \widehat{M}$.[1] For instance, $\mathbb{R}$ with its standard metric is a completion of $\mathbb{Q}$ equipped with the metric $d(x, y) = |x - y|$ (you might prove this in your writing assignment).

Your writing assignment is to:

(1) Define carefully complete metric spaces and the completion of a metric space, with some general introduction.
(2) Explore some properties of completions;
(3) Show by construction that every metric space M has a completion $\widehat{M}$; (this is one of the more involved parts of the assignment)
(4) Important examples and further directions;

Items 1 and 2 are background, and should be relatively short. The most important point that you should explain carefully is *Item 3*; here you should try to give proofs as carefully, and in as structured a fashion, as possible.[2] Item 4, examples, is the second most important part, and should feature a series of examples ranging from simple to more advanced in complexity.

You should think of this assignment that of writing a chapter in a book; you will want to give explanations and make sure to amply separate involved proofs by Lemmata, Propositions, Claims, etc. Note that Item 3 in particular appears in §46 of Johnsonbaugh and Pfaffenberger (see Theorem 46.7), which makes for a good reference. However, the proof given is very long and not, perhaps, amply explained or broken up into portions. Part of your assignment is to carefully explain this item, breaking the various stages of the proof down into Lemmas, etc.

2. Typsetting

As mentioned earlier, *it is required you typeset your paper*, and *we strongly recommend you use LaTeX*. We have a number of TeX templates available for you to use on the class

[1] Note that your textbook defines a *completion* of a metric space M to be any complete metric space N equipped with an isometric injection $i : M \to N$ with no need for it to be dense. For your writing assignment, you should use the definition given above, which seems to be more standard in the literature.

[2] Note: on HW3, you explored properties of *pseudometrics*; and any results you prove there you are welcome to cite, with a careful statement of the result.

webpage. If you have other questions about LaTeX, feel free to ask one of the course staff.

3. Due Dates

- The first draft is due **Friday, May 20** at 4 pm to Alex. (After this submission, you will receive by Monday, May 23, feedback from Alex on your draft).
- The final draft is due **Tuesday, May 31** at 4 pm to Alex.

For either submittion, you can e-mail your solutions if you would like.

4. On expository writing

The aim of the assignment is for you to practice writing mathematical *exposition*; it is important that your writing be precise yet understandable to someone else in your class with a comparable background. *Understandability* means several things:

- From a mechanical standpoint, you should avoid the use of any mathematical shorthand symbols like $\forall$ and $\exists$; intead write "for all" and "there exists." Always write in complete sentences, and explain conclusions to arguments.
- From a writing standpoint, you may choose first to give conceptual explanations of a fact before giving a rigorous proof. Or going even further, at times you may choose to omit a certain (inessential) proof, but only give a conceptual explanation of it.

5. Some more details

(1) **Define carefully complete metric spaces and the completion of a metric space.** Not too much to say here at the moment. Set up the topic of the rest of the paper, and give a very brief overview of what you will be covering.

(2) **Properties of completions** for instance, under what criteria does a continuous function $f : M \to N$ extends to a continuous function from the completion $\widehat{M} \to N$, and when is this extension unique (see exercise 46.12 of the book; f needs to be uniformly continuous[3])? Is the completion, when it exists, unique, in the following sense? If $(\widehat{M'}, i' : M \to \widehat{M'})$ is another completion of M, then there is an isometry $f : \widehat{M} \cong \widehat{M'}$ compatible with the inclusions of M, meaning that $f \circ i = i'$.

(3) **Construct the completion of a metric space:** The rough idea of this construction is that the set of Cauchy sequences in M, which we call $\tilde{M}$, has a distance function $d(\{a^{(k)}\}, \{b^{(l)}\}) = \lim_k d(a^{(k)}, b^{(k)})$ (check that this limit exists, and note that this check uses the completeness of the metric space $\mathbb{R}$!). However, this distance function is not a metric, but rather a *pseudometric* in the sense of Homework 3. By Homework 3, there is an induced quotient space $\tilde{M}^*$, equipped with a genuine metric d, which we call $\widehat{M}$. Then, one must verify that

(a) $\widehat{M}$ is complete, and

[3]One key intermediate Lemma, which may be useful to articulate/prove is: *if f is uniformly continuous, then the image of any Cauchy sequence is Cauchy.* It is easy to find examples, for instance $f(x) = 1/x$ on $(0, 1)$ of non-uniformly continuous functions which do not send Cauchy sequences to Cauchy sequences.

(b) there is a dense isometric injection of M into $\widehat{M}$.

(4) **Study some important examples of completions**: Choose some examples to discuss, such as the following: You should by no means discuss all of the below examples; this is just an incomplete list of ideas of possible topics to explore. **Ideally your study will have at least one more involved example, such as a discussion of options (a), (g), or (e).**

(a) Show that $\mathbb{R}$ is a completion of $\mathbb{Q}$. However, the main Theorem you prove cannot quite be used to give a construction of $\mathbb{R}$ "from scratch," at least as a metric space; this is circuluar! (note that metric is by definition a function to $[0, \infty) \subset \mathbb{R}$. Moreover, your construction of the pseudometric on the set $\bar{M}$ of Cauchy sequences used completeness of $\mathbb{R}$). But, one still has a "distance function" $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}_{\geq 0}$, and one consider the set $\widehat{\mathbb{Q}}$ of Cauchy sequences with respect to this function, modulo identifying Cauchy sequences whose limiting distance is zero. Instead of endowing the resulting set with a metric (which one cannot do yet if one has not constructed $\mathbb{R}$ or verified $\mathbb{R}$ is complete), one can then directly prove that the result satisfies the axioms of the real numbers; it's a field, the least upper bound property, etc.. Then, one just defines $d(x, y) := |x - y|$. You could explore this, or just indicate that some care/minor adjustments need to be taken to use Cauchy sequences to actually construct $\mathbb{R}$ (in order to not end up with a circular construction).

(b) the completion of $(0, 1) \subset \mathbb{R}$, equipped with the relatve topology.

(c) The examples in exercise 46.14, and various minor generalizations;

(d) Given a subset of $X \subset \mathbb{R}$ equipped with the relatve metric, any completion of X is isometric to its closure $\bar{X}$ inside $\mathbb{R}$. Is this also true for subsets $X \subset \mathbb{Q}$?

(e) Given a prime number $p \in \mathbb{N}$, there is a different metric that one can put on the rational numbers called the *p-adic metric* d_p: One defines

$$d_p(x, y) := \{p^{-e} | x - y = p^e \frac{a}{b} \text{ with } a,\ b, \text{ and } p \text{ relatively prime}\}.$$

(*Note*: the p-adic metric on $\mathbb{Q}$ is not isometric or even homeomorphic to $\mathbb{Q}$ with its usual metric.) For instance, the 5-adic difference between 64 and 39 is $\frac{1}{25}$, the 7-adic difference between 3 and 2 is 1, and the 3-adic difference between 1 and 5/9 is 9.

Show

(a) that this is a metric on $\mathbb{Q}$, and hence has a completion, called the *p-adic numbers* $\mathbb{Q}_p$.

(b) Just as a real number can be more simply be characterized in terms of their decimal expansions instead of as Cauchy sequences of rational numbers, elements in $\mathbb{Q}_p$ can also be characterized in terms of certain expansions: an element $x \in \mathbb{Q}_p$ can be written as

$$x = \sum_{k=-n}^{\infty} a_k p^k \text{ where } n \in \mathbb{N} \text{ and } a_k \in \mathbb{Z}.$$

Any other properties you may wish to examine about the p-adic numbers; there are many resources which one could google online.

Remark: The p-adic numbers may seem like an oddity, but they have become especially prominent in number theory and algebraic geometry. The metric spaces $\mathbb{Q}_p$ have a number of particularly nice properties: in addition to the ones above, $\mathbb{Q}_p$ is in fact a field *in which one can add, subtract, multiply, divide, etc.*

(f) Study the relationship between compactness and completion. For instance, we will show in class on Monday that every compact metric space is complete. There is a corresponding notion for non-complete spaces called *precompact*; a metric space is precompact if every sequence has a Cauchy subsequence.

(g) Show that that when a *normed vector space* is completed, the resulting completed metric space can also be given the structure of a *normed vector space* (this is essentially exercise 69.11). This would involve in part giving the definition/some examples of a normed vector space, something we have not done so far in class (but you can see the definition in §69).