1) Definitions / motivations / a first glimpse:

* \((X,\omega)\) symplectic manifold, \(L^n \subseteq X\) Lagrangian if \(\omega|_L = 0\).
* \(H : X \to \mathbb{R}\) Hamiltonian \(\sim X \ni \omega = \partial H\)
  \(\sim \phi^s_H\) time-\(s\) flow.

We allow time-dependent Hamiltonians \(H : (S^1 \times [0,\beta]) \times X \to \mathbb{R}\).

2) Lagrangian Floer homology:

- Impressionistically, it associated to a pair \((L_0, L_1)\) of Lagrangians a group \(HF^*(L_0, L_1)\) satisfying formally:

- Category: intersection \(\#^* : \chi(\text{HF}^*(L_0, L_1)) = L_0 \cdot L_1 \quad \text{smooth topology}\)
- Hamiltonian isotopy invariant: \(\text{HF}^*(\phi^*_H L_0, L_1) = \text{HF}^*(L_0, 
\phi^*_H L_1)\).
- If \(L_0 \not\approx L_1\), arrange that \(\text{HF}^*(L_0, L_1) = H^*(\text{CF}^*(L_0, L_1), S)\).

\[\phi^*_H L_0 \cap L_1 \geq \text{rk } \text{HF}^*(L_0, L_1) > L_0 \cdot L_1\]

So, \(\text{rk } \text{HF}^*\) gives a refined lower bound for Lagrangian intersections.

(i) \(\text{ex. } L : S^1 \subseteq C\). Note \(3 H : C \to \mathbb{R} : (x+iy) \mapsto 3y, \text{ with } X_H = -3 \partial_x\)

such that \(\phi^*_H (L) \cap L = \emptyset\): "\(L\) is displaceable."

So, if \(\text{HF}^*(L, L)\) existed and satisfied the properties above, we
would have \(\text{HF}^*(L, L) = 0\).
(ii) Say $\pi_2(x, L) = 0$. Floer proved that $HF^*(L, L)$ is defined, and $z \in H^*_{sing}(L)$. By (i), it can not be displaceable.

(iii) "Everything is a Lagrangian" (Weinstein)

Famous conjecture of Arnold: if $H : S^1 \times X \to \mathbb{R}$ generic (fixed points of $\phi_t^H$ isolated), then $\# Fix(\phi_t^H) \geq \text{rk } H^*(X)$ (lower bound fixed points).

Observe, given any $\phi : X \to X$ symplectomorphism, $\Gamma_\phi \subseteq X \times X$ is Lagrangian, and $\Delta \cap \Gamma_\phi \leftrightarrow \text{Fix}(\phi)$.

Arnold's conjecture would follow by showing $\text{rk } HF(\Delta, \Gamma_\phi) = \text{rk } H^*(\Delta) = \text{rk } H^*(X)$, but there are more direct methods.

(iv) There are cases where we can define $CF^*(L_0, L_1) \to S$, but $S \neq 0$.

Ex: $\square>L_0$. Following Floer, we say that $(L_0, L_1)$ is obstructed, or $L$ is obstructed if $L_0 = L_1 = L$.

Notes: we'll eventually need to clarify some issues:

- $HF^*(L_0, L_1)$ is often At first, $\mathbb{Z}$ and $\mathbb{Z}$, but grading? field?
- By choosing extra data on $L_0$ and $L_1$, we can often lift it to $\mathbb{Z}$ and $\mathbb{C}$, or $\mathbb{Z}$ and $\Lambda$ (Novikov field, needed for convergence issues).

To define $S$, we will study spaces of $J$-holomorphic discs in $M$, for some (auxiliary) almost complex structure $J$ on $M$.

1) b. Fukaya categories keep track of relationship between $HF^*(L_0, L_1)$ for varying $L_0, L_1$.

[Donaldson]: he observed that there is a "composition" $[\mathbb{Z}]: HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2)$ using the ambient geometry.
Thus, $L_i$ are objects of a category: the Donaldson-Fukaya category $H^0F$

**Objects:** $L_i \in X$ Lagrangians (unobstructed)

**Morphisms:** $\text{Hom}(L_i, L_j) = H^0F(L_i, L_j)$ (check: identity morphisms)

Unfortunately, this is insufficient for many purposes, e.g. for building/iterating LES in $H^0F(-, -)$. Indeed, we work at the chain level: $\text{hom}(L_0, L_\ast) := CF^\ast(L_0, L_\ast) \cong \mathbb{S} = \mathbb{S}^0$, and we have $p^2 : CF^\ast(L_0, L_k) \circ CF^\ast(L_0, L_l) \to CF^\ast(L_0, L_0)$.

**Problem:** $p^2$ is not associative.

Instead, the associator $p^3(-, p^3(-, -)) = p^3(p^3(-, -), -) = p^3(p^3(-), -) + \cdots$, i.e. it is chain homotopic to zero, for a chain homotopy $p^3$.

[Fukaya]: there is a hierarchy $p^k : CF^k(L_{k-1}, L_k) \circ \cdots \circ CF^0(L_0, L_k) \to CF^k(L_0, L_k)$ satisfying $\alpha = \sum_{i,j}(-)^{i+j}p^i(x_i, \cdots, x_{ij}, p^j(x_{i+1}, \cdots, x_i), x_i, \cdots, x_n) \forall \alpha$.

The first three are $(p^3)^2 = 0$, $p^3$ chain map, $p^3$ as above.

**Claim:** $(F(M), (p^i))$ An category is a quasi-isomorphism invariant.

$(F(M)$ is the right setting to talk about relations between Lagrangians (such as exact triangles, etc). Or rather, take the split-closed derived category $D^+F(M)$, whose objects are $\{L_0 \to L_1 \to \cdots \to L_k\}$.

ex: $L_2 \cong \{L_0 \to L_1\}$ means LES $\forall K : H^0F(L_2, K) \to H^0F(L_0, K)$

Rem: $D^+F(M)$ is related to mirror symmetry, a series of duality between symplectic geometry of $(X, \omega)$ and complex geometry of $(\overline{X}, \overline{J})$, discovered in string theory.
Conjecture: [Kontsevich] HMS: for each \((x, \tilde{x})\),

\[ D^* F (x, \omega) \cong D^b \text{Coh} (\tilde{x}, J) \]

Lagrangians, coherent sheaves, built out of hol. vector bundles and holomorphic submanifolds.

and moreover Kontsevich conjectured that this recovers \((*)\).

1) Flavors of Fukaya categories & their relations (useful in computing ordinary Fukaya categories!)

If \( M \) is non-compact, we might try to put non-compact LSM in \( F(M) \); there are then several choices for how to define the category:

(i) "small asymmetric Ham. perturbations near \( \omega \): \( \text{Ham}(K_0, \omega) \) = CF \( \phi_{\epsilon} (K_0, \omega) \)"

\[ \text{Co "infinitesimal Fukaya category"} \]

(ii) "arbitrarily large perturbations": \[ \text{Co "wrapped Fukaya category"} \]

(iii) Fukaya categories of singularities: \( F(E, W) \) for \( W : E \rightarrow C \)

Symplectic fibration with singular points (ex: Lefschetz fibrations, ie holomorphic Morse fibrations). This \( W \) specifies the non-compactness direction.

Rem.: \( D^* F(E, W) \) arises in HMS as mirror to \( D^b \text{Coh} (\tilde{x}) \), for \( \tilde{x} \) non Calabi-Yau: \( c_i (\tilde{x}) \neq 0 \).

\( D^* F(E, W) \) is an invariant of the singularities of \( W \).

ex: \( C^n \), \( W \) polynomial.
Meta-statements: [Seidel, Abouzaid- Seidel, Abouzaid- Ganatra]

1) In many cases, $F(E,W)$ is easier to compute than $F(E)$ (functors, exact sequences, etc).

2) $M = W^{-1}(p)$, there are relations between $F(E,W)$ and $F(M)$, allowing us to compute $F(M)$.

   - Study monodromy $N : M \to M$

3) Can use $F(E,W)$ and $F(E)$ as stepping stones to understand $F(E)$ and $W(E)$ (the wrapped Fukaya category).