Meta-statements: [Seidel, Abouzaid-Seidel, Abouzaid-Ganatra]

1) In many cases, $\mathcal{F}(E,W)$ is easier to compute than $\mathcal{F}(E)$ (functors, exact sequences, etc).

2) $M := W^{-1}(p)$, there are relations between $\mathcal{F}(E,W)$ and $\mathcal{F}(M)$, allowing us to
- compute $\mathcal{F}(M)$
- study monodromy $\mu : M \circlearrowleft M$

3) Can use $\mathcal{F}(E,W)$ and $\mathcal{F}(E)$ as stepping stones to understand $\mathcal{F}(E)$ and $\mathcal{W}(E)$ (the wrapped Fukaya category).

30/03/16

J-holomorphic curves & Lagrangian Floer homology, I.

Recall: (X^{an}, ω) symplectique.

Definitions: an almost complex structure J is $J \in End(TM)$ s.t. $J^2 = -\mathbb{1}$.

J is compatible with ω if $\omega(-, J-)$ is a metric.

J is tamed by ω if $\omega(v, Jv) > 0$ if $v \neq 0$ ($\Rightarrow \frac{\omega(v,Jw)+\omega(Jv,w)}{2}$ is a metric).
↳ open condition.

Rem: the entire theory we'll develop can be done for tame a.c.s. It is sometimes handy, as there exists more tame a.c.s.

ex: for (X,ω) a Kähler manifold, $J : TX \circlearrowleft$ induced is an integrable structure.

J-holomorphic curve: fix (X,J) almost complex manifold, (Σ, j) a Riemann surface (possibly with ∂, and with standard complex structure).

Definition: $u : \Sigma \to X$ is J-holomorphic if $du \circ j = J \circ du$
$\Leftrightarrow \bar\partial_J u := \frac{1}{2}(du + J \circ du \circ j) = 0$. In local coordinates on Σ, $s+it$, this is equivalent to the usual Cauchy-Riemann equations $\partial_t u = J \partial_s u$.

(*)

Given a metric g (and a fixed one on Σ), get a metric on $\text{Maps}(T\Sigma, u^*TX)$, so we can define the <u>energy</u> of such a map: $E(u) := \int_\Sigma |du|^2$.

Proposition (identity energy): if ω is symplectic, J compatible, $g := \omega(-, J-)$ and u is J-holomorphic, then

$$E(u) = \int_\Sigma u^*\omega \qquad \text{(exercise).}$$

↳ only depends on homotopy class of u ⇒ a priori controled.

⊛ Rem: we might impose Lagrangian boundary conditions, namely $u|_{\partial\Sigma}$ maps into Lagrangian submanifolds.

Towards Lagrangian Floer homology:

Let $L_0^n, L_1^n \subseteq (X, \omega)$ Lagrangian submanifolds.

The Floer homology, formally, will be the Morse homology theory for a symplectic action functional on $\mathcal{P}(L_0, L_1) = \{\gamma : [0,1] \to X \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}$.

↳ "$\mathcal{A} : \mathcal{P}(L_0, L_1) \to \mathbb{R}$".

Recap: if $f : M^k \to \mathbb{R}$ is a Morse function on M (smooth, finite dimensional), we get $HM^*(f) = H^*(CM^*(f), \delta)$ where $CM^*(f) = \bigoplus_{p \in \text{crit}(f)} \mathbb{C}\langle p\rangle$ and δ counts flowlines of ∇f (w.r.t. some g) between critical points p and q. A flowline is a map $\gamma : \mathbb{R} \to M$ st $\dot\gamma = \nabla f$, $\lim_{s\to-\infty} \gamma(s) = p$, $\lim_{s\to+\infty} \gamma(s) = q$.

Actually, it turns out that in general, only $d\mathcal{A}$ is well-defined (which is good enough to deal with $\nabla \mathcal{A}$). We have

$$\mathcal{A} : \tilde{\mathcal{P}}(L_0, L_1) \to \mathbb{R} : (\gamma, [u]) \mapsto \int u^*\omega$$

↳ univ. cover: elements are $(\gamma, [u])$, where $u : [0,1]^2 \to X$ is a path in $\mathcal{P}(L_0, L_1)$ from $*$ (a base point) to γ.

↳ we assume $\mathcal{P}(L_0, L_1)$ is connected.