01/04/16

Last time: $L_0, L_1 \subseteq (X^{2n}, \omega)$ Lagrangian submanifolds, $L_0 \pitchfork L_1$.

Fix Λ field and $T \in \Lambda$:

- realistically, $\Lambda = \{\sum a_i T^{\lambda_i} \mid \substack{\lambda_i \to \infty,\ \lambda_i \in \mathbb{R} \\ a_i \in k = \mathbb{C}}\}$, T the formal variable
- dream (+ some nice situations): $\Lambda = \mathbb{C}$, $T = 1$.

We tentitavely defined, based on the idea of Morse theory for $\mathcal{A}: \overset{(\sim)}{\mathcal{P}}(L_0, L_1) \to \mathbb{R}$, $CF^*(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$ and, wrt some almost complex structure J on X, a "differential"

$$\partial p = \sum_{\substack{q,\, u \in \mathcal{M}(p,q,J)/\mathbb{R} \\ \text{rigid}}} T^{\omega(u)} \cdot \mathrm{sgn}(u) \cdot q ,$$

$\mathrm{sgn}(u)$: ± 1 (imagine $\Lambda = \mathbb{Z}_2$, so $+1$).

where $\mathcal{M}(p,q,J) = \coprod_\beta \mathcal{M}(p,q,\beta,J)$, $\beta \in \pi_2(M; L_0, L_1, p, q)$

and $\mathcal{M}(p,q,\beta,J) = \left\{ \begin{array}{l} u: (\mathbb{R}\times[0,1], j) \to (X, J) \text{ in class } \beta \\ u(s,i) \in L_i \text{ for } i \in \{0,1\} \\ \lim_{s\to+\infty} u(s,t) = p, \quad \lim_{s\to-\infty} u(s,t) = q \\ \bar\partial_J u = \frac{1}{2}(du + J\circ du \circ j) = 0 \\ E(u) = \int u^*\omega < \infty \end{array} \right\}$

Want: (a) For generic J, each $\mathcal{M}(p,q,\beta,J)$ is a finite-dimensional manifold, with a fixed "expected dimension" $=: \mathrm{ind}_{\bar\partial_J}(\beta)$.

(b) $\mathcal{M}(p,q,\beta,J)$ compact(ifiable) with the desired "boundary strata"

(c) If the above hold, δ is well-defined and $\delta^2 = 0$.

Let's talk about (a). Last time, we observed we could express $\mathcal{M}(p,q,\beta,J) = \bar\partial_J^{-1}(0)$ where $\bar\partial_J$ is a section of an infinite-dimensional vector bundle $\begin{array}{c}\mathcal{E}\\ \downarrow \pi \\ \mathcal{B}\end{array}$, $\bar\partial_J$, for

$\mathcal{B} = C^\infty(\mathbb{R}\times[0,1], X, L_0, L_1, p, q, \beta)$

$\mathcal{E}_u = C^\infty(\underbrace{\mathbb{R}\times[0,1]}_{=:S}, \Omega^{0,1}_S \otimes u^*TX)$.

Toy model from finite dimensional differential topology: if $f: M^m \overset{C^\infty}{\to} N^n$ submersion at $p \in M$ (ie $Df_p: T_pM \twoheadrightarrow T_{f(p)}N$), the implicit function theorem tells us that $f^{-1}(q)$ near p has a
$q = f(p)$

smooth manifold structure, of dimension $m-n$.

Somewhat closer, V rank k vector bundle, want $s^{-1}(0)$ to be a smooth manifold of the "right dimension" $(m-k) \Leftarrow s(M) \pitchfork$ 0-section in V.

$$\begin{tikzcd} V \arrow[d, "\pi"] \\ M^m \arrow[u, bend left, "s"] \end{tikzcd}$$

The implicit function theorem applies at $p \in s^{-1}(0)$, provided we check $d\check{s} : T_p M \longrightarrow T^{vert}_{s(p)} V := \ker d\pi$ is surjective.

Moreover, $T_p(s^{-1}(0)) = \ker d\check{s}$.

Even if $d\check{s}$ is not surjective at p, we can determine an "expected dimension" as $\operatorname{ind}(d_p\check{s}) := \operatorname{rk} \ker d_p\check{s} - \operatorname{rk} \operatorname{coker} d_p\check{s}$.

(it is $m-k$ here, independent of p).

For us, M and V are infinite dimensional, but in the <u>nicest possible</u> <u>way</u>:

<u>Theorem</u> [Floer]: if $L_0 \pitchfork L_1$, after extending $\bar\partial_J$ to a suitable Sobolev completion, solving $\bar\partial_J u = 0$ is a Fredholm problem, meaning that we have

$$\begin{tikzcd} \mathcal{E}^{k-1,p} \arrow[d, "\pi"] & \text{(Banach bundle)}, \\ \mathcal{B}^{k,p} \arrow[u, bend left, "\bar\partial_J"] & \text{(Banach manifold)} \end{tikzcd}$$

such that the linearization (vertical part of $D(\bar\partial_J)$) at $u \in \bar\partial_J^{-1}(0)$

$$D^u_{\bar\partial_J} : W^{k,p}(S, u^*TX, u^*TL_0, u^*TL_1) \longrightarrow W^{k-1,p}(S, u^*TX)$$

is Fredholm, i.e.
- the image is closed
- $\operatorname{ind}(D^u_{\bar\partial_J}) = \operatorname{rk}\ker(D^u_{\bar\partial_J}) - \operatorname{rk}\operatorname{coker}(D^u_{\bar\partial_J}) < \infty$.

We say that u is regular is $D^u_{\bar\partial_J}$ is surjective. In this case, an infinite-dimensional version of the implicit function theorem says that near u, $M(p,q)$ is a finite dimensional manifold of