Today: remarks on stable maps

- Floer's equation vs $S^1$-hol strips
- Floer's thm: $\text{HF}^+(L_0, L) \cong H^*(L_0)$
- Oh spectral sequence

**Stable maps:** A $S^1$-hol map $u: \Sigma_\infty \to X$ is called a pseudo stable map; it is stable if its automorphism group is finite, where a

automorphism is $\Sigma_\infty \xrightarrow{f} \Sigma_\infty$ where $f$ is the underlying automorphism

$\Sigma_\infty \xrightarrow{\phi}$ of trees + birational on components, preserve marked points.

**Remark:** if $u: \Sigma_\infty \to X$ is constant, then stability of $u$ implies stability of the domain $\Sigma_\infty$ (equipped with marked points). If $u$ is constant, then $\Sigma_\infty$ is stable $\Rightarrow$ $\Sigma_\infty$ is stable.

**Example:** $u: L_\infty \to X$ non-constant is stable, because any automorphism $\phi: (\mathbb{R} \times \mathbb{R}, 0, 0)$ is translation, and $u \circ \phi = u$.

**Remark:** in Gromov's compactness, bubbles are only well defined up to overall automorphism $\Rightarrow$ get an element of $\text{HF}^+$ of domains.

"At of domains" has a nearly free action when the maps are stable.

**Floer's equation:**

It is helpful to have a more flexible way of constructing Floer homology groups, using an auxiliary Hamiltonian.

Define, for $H: L_0, L \to \mathbb{R}$ Hamiltonian, Jacobi $\mathbb{R}$ and $L_0$ and $L_1$, $\text{CF}^+(L_0, L_1, H)$, defined whenever $\partial^+_H(L_0) \cap L_1$.

Let $X^{H^+}_{L_0, L_1} = \{\text{time-1 chords of } X_H \text{ from } L_0 \text{ to } L_1\}$

$g: L_0, L_1 \to X$, $g(0) \in L_0$ for $c \in \mathbb{R}$, $g = X_H$. 

Define $CF^*(L_0, L_1; H, \mathbf{J}) = \prod_{n=1}^{\infty} X_{L_0, L_1}^n$.

Differential: for $x^+ \in X_{L_0, L_1}^n$, $\beta \in \pi_2(x^+, x^-, \ldots)$,

$$
\begin{aligned}
\mathcal{M}(x^+, \tilde{x}, \beta) &= \left\{ u : (S^1, 0, 1) \to X \mid
\begin{array}{l}
\mu : (S^1, 0, 1) \to (S^1, 0, 1) \\
u(0, 1) \in L \\
\partial_s u + J(d \mu - X_\mu) = 0
\end{array}
\right\}
\end{aligned}
$$

Floer's equation

$$
E(u) = \int_S |\partial \mu - x|^2 + \int_S (x \cdot \omega - d(x^* \mu dt)) < \infty.
$$

Coordinate free Floer's equation: $\left( d\mu - X_{\mu} \otimes dt \right)_{\mathbf{J} + J} = 0$

where $f \in Hom(TS, \omega^* TX)$, have $f_{\mathbf{J} + J} = \frac{1}{2} \left( f + J f_{\mathbf{J}} \right)$.

There are manifolds of index $\text{ind}(\beta)$ for generic $J$; look at $\mathcal{M}(x^+, \tilde{x}, \beta)/\mathbb{R}$ when $\text{ind}(\beta) > 0$. Then Gromov compactify.

Define $d(x^+) = \sum_{\mathbf{J} \in \pi_2(x^+, x^-)} e_{\mathbf{J}} \# \left( \mathcal{M}(x^+, \tilde{x}, \beta)/\mathbb{R} \right)$, $x^-$: Gromov compactness and gluing imply $d^2 = 0$ (in absence of bad bubbles).

Note: given $\mu \in \mathcal{M}(x^+, \tilde{x})$, gauge transform $\tilde{\mu}(s, t) = \phi_{H_t}^N \mu(s, t)$.

Then, $\tilde{\mu}$ is a solution to $\tilde{\partial}_J \tilde{u} = 0$ with boundary on $\phi_{H_t}(L_0, L_1)$, asymptotic to $x^+(n)$, $x(n)$, where $J = (\phi_{H_t})^* J (\phi_{H_t})^{-1}$ depends on $t$ (in general, we may have needed $t$-dependent $J$ anyway for transversality).

$\Rightarrow \exists \mathbb{C}F^*(L_0, L_1; H, \mathbf{J}) \cong \mathbb{C} F^*(\phi_{H_t}(L_0, L_1), L_1; H, \mathbf{J})$ as chain complexes.

As an example of application, we look at continuation maps $\mathbb{C}F^*(L_0, L_1; H, \mathbf{J}) \cong \mathbb{C}F^*(L_0, L_1, H, \mathbf{J})$. If there are quasi-isomorphisms, then $HF^* (\phi_{H_t}(L_0, L_1), L_1) \cong HF^*(L_0, L_1)$.

Define $\tilde{\partial}_{\mathbf{J}} u = \left( d\mu - x_\mu \otimes dt \right)_{\mathbf{J}}$ for $\mu$. 

Count index of solutions to

\[ x^+ \text{ bubble of } X, \quad \bar{\partial}^L \bar{H} = 0 \]

\[ \bar{\partial}^L \bar{H} = 0 \]

Chain map (if no bad bubbling)

\[ \Phi \] follows from Gromov compactness/gluing, applied to index 1 moduli space. Namely,

\[ \Phi \text{ (index 1 moduli space) is} \]

\[ \begin{pmatrix} H_1 & H_0 \end{pmatrix} \begin{pmatrix} H_0 \to R \end{pmatrix} = \begin{pmatrix} H_0 \to H_0 \end{pmatrix} \]

\[ \Phi \text{ on } \Psi_{H_0, H_0} \circ \Psi_{H_0, H_1} \text{ counts } \]

\[ \sum_{\text{ind}(\rho) = 0} T_{\rho} \quad \# \lambda H_{H_0, H_1}(x^+) \]

Remark  such a map can have negative energy, namely

\[ E(\mu) = \int \mu^* \omega - d(\mu^* \lambda dt) \]

now equals

\[ \int \omega_{H_0, H_1} \lambda H_0, H_1 \lim \to 0 \text{ bounded, indep of } \mu \text{ so don't get powers too negative, (x)} \]

\[ (d \text{ vert } \mu^* \lambda dt + d \text{ horiz } \mu^* \lambda dt = \partial \text{ H d s d t) } \]

In general, the continuation maps involve negative powers of \( T(\mu) \), invariance requires Novikov field, not ring.

\[ \text{Co powers bounded below, ok by (x)} \]

\[ \Psi_{H_0, H_0} \circ \Psi_{H_0, H_1} \text{ counts } \]

\[ \Psi_{H_0, H_0} \circ \Psi_{H_0, H_1} \]

"Homology of homotopies" to show \( \Psi_{H_0, H_0} \circ \Psi_{H_0, H_1} = \text{id} = 8K + K \partial \),

let \( H_{H_0, H_1} \) denote the space of pairs \( (\lambda \in C_{0,0}) \text{, in solution} \)

\( b \text{ to } x^+ \)

If \( \lambda < 1 \), then \( H_0 \to \]

If \( \lambda > 2 \), then \( H_0 \to \]

\( \lambda \text{ to } H_0 \to \]

\( R \text{, } \lambda \to \)

\( R \text{, } \lambda \to \)
A count of index 0 elements of $\mathcal{M}_{\lambda, H_0}$ will define a map $K: CF^*(L_0, L_0, H_0) \to CF^*(L_0, L_0, H_0)$. Gromov compactness and gluing imply (if no bad bubbling) that there are 4 types of phenomena that can happen on $\partial$ (index $\lambda$ moduli space):

1. $\lambda \to +\infty$: $\quad H_0 \quad \lambda \to +\infty \quad H_0$

2. $\quad (\quad H_0 \quad \lambda \to +\infty \quad H_0 \quad )$

3. $\lambda \to -\infty$: $\quad H_0 \quad \lambda \to -\infty \quad H_0$

4. $\lambda \to 0$: $\quad x \quad H_0 \quad x^+$. But: index 0 solutions (not $\lambda R$).

Claim: the index 0 elements of $\mathcal{M}(x^+, x^-)$ are exactly constant maps ($\Rightarrow$ count gives $\text{Id}$).

So, mod signs, get $K \circ \partial H_0 - \partial H_0 \circ K + \text{Id} = \Phi_{H_0, H_0} \circ \Phi_{H_0, H_0} = 0$.

The reverse composition is identical computations, so we get that $\Phi_{H_0, H_0}$ is a homology iso.

Rem: Invariance $\Rightarrow$ define $HF^*(L, L) = HF^*(L, L, H, J)$ for $H$ with $\partial H(L) = L$. Invariance: for $J$ is similar.

Ex: $X = T^*\mathbb{Q}$, $\omega_{can} = d\lambda_{can} = dp \wedge dq$, $L \leq X$ $\omega$-section.

$L \leq X$ is exact, meaning $\lambda_{can}(L) = 0$ (in fact, $\lambda_{can}(L) = 0$).

In particular, Stokes $\Rightarrow$ $\pi_2(T^*\mathbb{Q}) \cong \pi_2(T^*\mathbb{Q}, L) = 0$.

So, $HF^*(L, L)$ is well-defined, as long as Gromov compactness holds. (Gromov compactness requires all strips to be mapped to $C \subset T^*\mathbb{Q}$ compact, but in principle, strips could escape to $\infty$ in target.)
Claim: have an a priori $C^0$ estimate on floor curves
\[ u: \mathbb{R} \times [0,1] \to (T^*Q, L). \]
It comes from "maximum principle", "monotonicity" (\( T^*Q\) is convex at \( \alpha\), Liouville,...)

Assuming the claim, we have \( \text{HF}^*(L, L) \)

**Theorem (Floor):** \( \text{HF}^*(L, L) \subseteq H^*(L) \)

(more generally true for any \( L \leq X\), provided \( \pi_2(X) = 0 = \pi_2(Y, L) \).

**Proof:** by invariance, compare \( \text{CF}^*(L, L; H, J)\) with \( \text{CF}^*(L; \text{CF}^*(f, g)\) for specific nice \( H, J\). Pick \( g \) near \( L \), \( f: L \to \mathbb{R}\) Morse function with \((f, g)\) "Morse-Smale".

Note: \( g\) on \( L = Q\) induces a splitting \( T^{*}Q \simeq T_{\text{vert}}^{*}Q \oplus T_{\text{vert}}^{*}Q\), and induces a \( \mathcal{I}\) on \( T^{*}Q\).

On \( T(T^{*}Q)|_Q \subseteq T^{*}Q \oplus Q\),
\( \mathcal{I}\) on \( q\) is the natural pairing \( \mathcal{I}\) induced by \( g\):
\[ \mathcal{I}(\phi) = g(\phi, -) \]
\[ T_Q^{*}Q \quad T_Q^{*}Q \]

Note: \( f: L \to \mathbb{R}\) induces a Hamiltonian \( H: T^*_Q \to \mathbb{R}\)
(maybe cutoff \( H\) near \( \infty\):)

**Theorem (Floor).** if \( f\) is \( C^2\) small, there is a bijection
\[ \{ g: \mathbb{R} \to Q, \ g(s) = \nabla f, (s) \} \leftrightarrow \left\{ \text{Solutions to Floer's equation} \right\} \]
\[ \{ \text{flowlines for } (f, g) \} \leftrightarrow \left\{ \text{Solutions to } H = \mathcal{I}(\phi, -) + \mathcal{I}(\phi, -) = 0 \right\} \]
with asymptotics \( x^3\)

Note: in coordinates, \( H(q, p) = f(q) \)
\[ dH = f_q(q) dq \]
\[ X_H = f'_q(q) dp \]
Note: \( X_4 = 0 \) at critical points of \( f: \mathcal{Q} \rightarrow \mathbb{R} \), so

\[
\{ \text{constant trajectories} \} = \text{crit}(f).
\]

Further: if \( y: \mathcal{R} \rightarrow \mathcal{Q} \) is st 
\[
y(s) = \nabla f(y(s)), \quad u(s,t) = y(s)
\]
satisfies
\[
\frac{\partial u}{\partial s} + \int_0^t \left( \frac{\partial u}{\partial s} - x \right) = 0 \quad y(s) = 3x = -\Phi f
\]

In general, for \( \pi_2(x, \pi_2(x, L) \neq 0 \), there may be discs/sphere classes, and in particular new classes of strips, by "connect sums" of homotopy classes.

If \( d \) happens to be well-defined and \( a^2 = 0 \), one can look at the energy filtration of terms contributing to \( d \).

Because

(a) Near any \( L \times X \), \( \mathcal{N} \) Weinstein nbhd \( U \subset T^*L \) of \( L \)
(b) "low energy strips/discs" must stay inside \( U \) (monotonicity lemma),

the low energy part of \( d \) coincides (for nice \( H, J \)) with the Morse differential, by Floer's argument.

\[ \Rightarrow \text{3 spectral sequence} \quad H^*(L) \Rightarrow HF^*(L, L) \quad [\text{Oh spectral sequence}] \]

More generally, if \( L_0, L_1 \) have clean intersection so that \( L_0 \cap L_1 : N \times X \), Pozniak constructed a spectral sequence (under hypotheses of definedness)

\[ H^*(N) \Rightarrow HF^*(L_0, L_1), \quad \text{coming from a reduction to a local model in low energy}: \quad \mathcal{Q} \subset T^*\mathcal{Q} \text{ zero section} \]

\[ L \subset \text{conormal to } N \]

\[ \nu^*(N) = \{ (q, p) \mid q \in N, p \text{ annihilates } TN \} \]