11/04/16

Today: 1) gradings
2) product structures
3) signs

Recall: $L_0, L_1 \subseteq (X, \omega)$, J a.c.s., $H: [0,1] \times X \to \mathbb{R}$ (possibly 0).

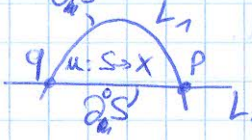
We defined $CF^*(L_0, L_1; H, J) := \Lambda^{|\phi_H^1 L_0 \cap L_1|}$ ← think of this as time-1 chords $L_0 \to L_1$ of X_H,

with $\delta p := \sum_{\substack{q,\, \beta \in \pi_2(p,q) \\ \mathrm{ind}\, \beta = 1}} T^{E(\beta)} \# \left(\mathcal{M}(p,q) / \mathbb{R} \right) \cdot q$.

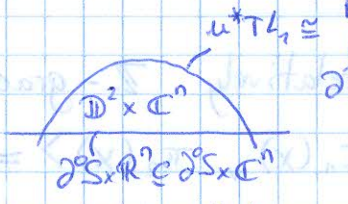
In nice cases, δ is well-defined, $\delta^2 = 0$, $HF^*(L_0, L_1)$ does not depend on H and J.

Gradings: (assume $H=0$ for simplicity)

For $[u] \in \pi_2(p,q) = \pi_2(X; L_0, L_1, p, q)$, we defined $\mathrm{ind}([u])$ as follows:



Simultaneously trivialize $u^* TX$ and $(u|_{\partial^0 S})^* TL_0$, to get:



So get $\partial^1 S \to \Lambda(n)$ Grassmannian of linear Lagrangians $\subseteq \mathbb{C}^n$,

e.g. a path γ_t in $\Lambda(n)$ which is transverse to $\mathbb{R}^n \subseteq \mathbb{C}^n$ at 0 and 1

↳ define $\mathrm{ind}([u]) := \gamma_t \cdot \underbrace{\Lambda_1(n)}_{:= \{ L \subseteq \mathbb{C}^n \text{ linear Lagr} \mid L \cap \mathbb{R}^n \neq \{0\} \}}$.

Want: assign an absolute $\mathbb{Z}$-grading to a $p \in L_0 \cap L_1$, called $\deg(p)$, so that $\deg(q) - \deg(p) = \mathrm{ind}([u])$ for any $u \in \pi_2(p,q)$.

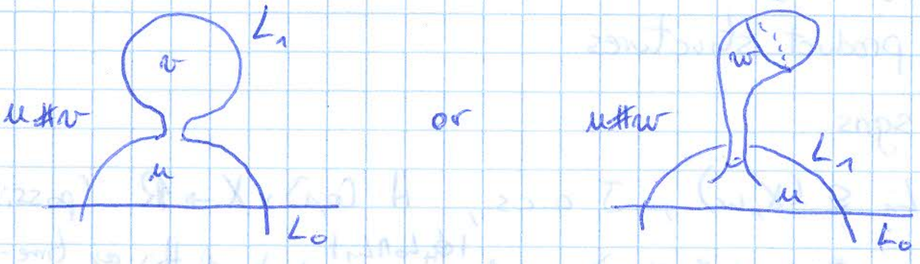
Of course, this is impossible if there are $\pi_2(p,q)$ classes whose indices differ: $[u], [v] \in \pi_2(p,q)$ s.t. $\mathrm{ind}([u]) \neq \mathrm{ind}([v])$.

Sources of ambiguity: in $\mathrm{ind}(\cdot)$:

Given $[u] \in \pi_2(p,q)$ and $v: (\mathbb{D}^2, S^1) \to (X, L_i)$ $\quad v \in \pi_2(X, L_i)$,

or $w: S^2 \to X$ $\quad w \in \pi_2(X)$

Connect sum give new elements of $\pi_2(p,q)$:



For disc: $\mathrm{ind}([u\#v]) = \mathrm{ind}(u) + \mu(v)$ (Maslov index)

For sphere: $\mathrm{ind}([u\#w]) = \mathrm{ind}(u) + 2\langle c_1(TX), w_*[S^2]\rangle$

So in general, we can only even associate relative gradings in $\mathbb{Z}/N\mathbb{Z}$, where $N\mathbb{Z}$ is the subgroup generated by these ambiguity terms $\mu(v)$ and $2\langle c_1(TX), w_*[S^2]\rangle$.

Exercise: if L_0, L_1 oriented, then $\mu(v) \in 2\mathbb{Z}$ (disc class in $\pi_2(X,L_i)$), so $2 \mid N$ and $CF^*(L_0,L_1)$ is relatively $\mathbb{Z}_2$-graded.

We can relatively $\mathbb{Z}$-grade if

(a) $\langle 2c_1(X), \pi_2(X)\rangle = 0$

(b) $\mu(v) = 0$. If (a), we can think of it as $\tilde{\mu}: H_1(L_i) \to \mathbb{Z}$, via $\mu(\rho \in \pi_2(X,L_i)) = \tilde{\mu}(\partial\rho)$. This is well-defined if (a), since two ρ's with same $\partial\rho$ differ by a sphere. We want $\tilde{\mu}$ to vanish.

Absolute gradings [Kontsevich, Seidel]

Fixing extra data on X, L_i, we can associate absolute $\mathbb{Z}$-gradings to $CF^*(L_0,L_1)$.

Idea: $\exists$ choices of paths between $\Lambda_0, \Lambda_1 \in \Lambda(n)$, but not between $\Lambda_0^\#, \Lambda_1^\# \in \tilde{\Lambda}(n)$ the universal cover.

Recall: Lagr. Grassmannian $\Lambda(n) \cong U(n)/O(n)$, $H^1(\Lambda(n);\mathbb{Z}) = \mathbb{Z}\langle\mu\rangle$ and $\pi_1(\Lambda(n)) \cong \mathbb{Z}$, with $\det^2: U(n) \to S^1$ a π_1-iso which classifies μ.