The equation \( \partial \omega + J \partial \omega = 0 \) 

Given \( x_1 \in \mathcal{C}^*(L_0, L_1, J_1) \), \( x_2 \in \mathcal{C}^*(L_0, L_2, J_2) \) and \( x_{ob} \in \mathcal{C}^*(L_0, L_2; J_2) \), get \( (\partial x_1, x_1, x_2) \) 

(make some choices of ends, \( J \) restricting to \( J_0, J_1, J_2 \) on various ends, etc.). A count of index 0 solutions will give matrix coefficients for the map \( L^p \) on chain level.

Before continuing talking about product structures, let's talk about signs...

**Orientations:** [Floer-Hofer (From Floer homology), deSiard, Fukaya (large setting), Seidel, Wehrheim-Woodward 'Orientations', Abouzaid's monograph] 

Want to associate a signed count \( \#(\mathcal{H}(p,q)/R) \) in \( 2: \mathcal{C}^*(L_0, L_1) \). 

More generally, we want an orientation of \( \mathcal{H}(p,q)/R \) or by trivializing the \( R \)-action, orient \( \mathcal{H}(p,q) \).

**Meta-statement:** (not the most general) \( \mathcal{H}(p,q) \) is canonically oriented, relative to certain orientations of 1-dim vector spaces associated to \( p, q \) (the orientation lines \( o_q, o_q \)), after fixing spin structures on \( L_0 \) and \( L_1 \).

\( L_0, L_1 \subseteq X \), \( p, q \in L_0 \setminus L_1 \), say \( L_0, L_1 \) oriented.

**Rem:** \( \text{spin}(n) \to \text{so}(n) \), is the universal cover if \( n \neq 3 \).

**Rem:** more generally, could ask \( L_i \) to be relatively \( \text{Pin} \), rel \( \text{bH}^2(X; \mathbb{Z}/2) \).

Spin corresponds to \( b = 0 \) and \( L_i \) oriented as well, condition \( b L = \text{w}_2(L) \). (rel b)

So, Floer cohomology with signs a grading has objects \( (L, \text{spin} \text{ structure}, \text{a} \text{ grading}, ...) \).

**Notation:** \( V \) finite dimensional vector space \( \Rightarrow \det(V) = V \text{ top } V \), \( \mathbb{Z}/2 \)-graded 1-dimensional vector space, grading is \( \dim(V) \).
For $M$ a manifold, get $\lambda(TM)$ the determinant line bundle. An orientation of $M \mapsto$ trivialization of $\lambda(TM)$. For $D: X \to Y$ a Fredholm operator, get $\det(D) = \lambda(\text{ker } D)^* \otimes \lambda(\text{coker } D)$, so get a natural line bundle $\det \to \text{Fred}(x, y)$.

More specific to our setting: $\pi^{-1}_X S_X$ Banach bundle, with $s$ a section whose linearization is Fredholm. So, get $\det$ and $\overline{\det}$, $\det \otimes \overline{\det} \cong \det(D(s))$.

Note: if $M - s^*(0)$ is transversely cut out, then $\det_M \cong \lambda(\ker D(s))$. So, an orientation/trivialization of $\det_M$ is the same as an orientation of $M$.

The orientation line of $p \in L_o \cap L_1$ (or more generally, $L_0 \subseteq L_1$, time $t$ chord of $X_H$): the idea is, given $p$, construct a Cauchy-Riemann operator $\partial p$ associated to $p$, and take $D_p := \det(\partial p)^*$. Fix $p \mapsto \text{a bundle pair } (E, F)$ negative strip-like end around $t_1$ (or $\infty$). $E$ is the trivial bundle $\mathbb{D}^2 \times T_p X \cong \mathbb{D}^2 \times \mathbb{C}$, and $F$ is a path of Lagrangian subspaces between $\Lambda_0 := T_p L_0$ and $\Lambda_1 := T_p L_1$, constant near $\infty$.

Rem: if the $L_i$'s come equipped with a grading structure, e.g. lifts $\tilde{L}_i \to L_i$, then $\Lambda_t$ is uniquely determined up to homotopy rel endpoints.
Choosing $J$ on $E$, we get a local Cauchy–Riemann operator $\tilde{\partial}(E, F)$, acting on sections which are asymptotic near $e_0$ to $N_0 \cap N_1$, i.e.

$\tilde{\partial} : W^{k,p}(H, E, F, 0 = N_0 \cap N_1) \to W^{k-1,p}(H, SL^0 E \otimes E)$.

Linearize $\tilde{\partial}$, get $D_p$ a Fredholm operator.

**Definition:** $\text{det}(D_p) = \det(D_p)$

**Claims:**
1. Independent (up to canonical isomorphism) of choice of $J$, etc.
2. An elaboration of index calculation (using $\text{ind}(D_p) = \#t$ times where $N_t$ (of $N$, with signs) shows that

\[
\text{det}(D_p) \cong \prod_{t \in \text{times that } N_t \cap N_t \text{ ordered}} V_{t_i},
\]

where $V_{t_i} = (N_{t_i} \cap N_{t_i}) \otimes \text{sign } N_t \cap N_t$

for generic $N_t$, this is $1$-dim.

**Theorem:** Let $E_0, L_1$ oriented, $E_0$ equipped with fixed spin structures.

Then, there is a canonical isomorphism (up to some positive rescaling)

\[
\Lambda (TM(p, q)) \cong \text{det}(D_p) \otimes \text{det}(D_q).
\]

How does this work, roughly? It's a consequence of gluing theorems for determinants of (local) Cauchy–Riemann operators.

Given $x \in \mathcal{M}(p, q) \times \mathcal{L}_0$, trivialize $T^* TX$ to obtain a local Cauchy–Riemann operator $D_x : S \times C^\infty \to N_t$, and linearize $D_x$.

Can glue:

\[
\text{det}(D_p) \otimes \text{det}(D_q) \cong \text{det}(D_p \otimes D_q),
\]

because it was constant near the strip-like end. Finally, homotope to $D_0$.

Gluing theorem: $\text{det}(D_p) \otimes \text{det}(D_q) \cong \text{det}(D_0 \otimes D_q)$ $R$-large, gluing together elements of ker/coker using cutoffs $\text{det}(D_q)$.
Need to check: independence of
(a) choice of trivialization of $u^* T X$
(b) choice of homotopy
(c) choice of path $\gamma$

Choose one $x_0$ in each connected component, connect up to an arbitrary $x \in X(p,q)$ by a path $\gamma : x_0 \rightarrow x$, and use the induced isomorphism from $x_0$.

Point: a Spin structure on $L$ determines a distinguished class of (stable) trivializations of $T L$ on $1$-skeleton which extends to the $2$-skeleton.

[Compare Theorem (de Silva, Hirasawa)] If $L$ is oriented, then a Spin structure on $L$ induces an orientation of $\mathcal{H}(x, L; J) = \{ u : (x, 0, s) \rightarrow (x, l) \mid \delta_J u = 0 \}$.

Floer theory with signs

There are 2 options:

(a) "Coherent orientations" (Floer–Hofer), requires some choices of

(b) "Canonical orientations" (Seidel).

We'll do (b), and then see how to get (a) from it.

Let $\text{CF}^+(L_0, L_1 ; H, J) := \bigoplus_{x \in L_0} \Lambda^{x_0, x_1}$, \( \Lambda \) - graded field.

Given $V$ a real $1$-dim vector space, $\text{IV}_{12}$ is the free $\mathbb{R}$-module generated by orientations of $V$ modulo the sum of opposite orientations vanishes.

Note: $\text{IV}_{12} \cong \mathbb{Z}/2$ canonically.
Given $u \in H(p,q) \otimes R$ rigid, $\lambda(T_u H(p,q) \otimes R) \cong R$. After trivializing the $R$-action, get

$$R \cong \lambda(T_u H(p,q)) \cong o_q \otimes o_p^{'},$$

so we get isomorphism $\mu_u \colon o_p \to o_q$, which induces

$$\mu_u \colon 1 o_p \to 1 o_q.$$

So, for $E \times \bar{E} \in 1 o_p 1_{\Lambda}$, define

$$d(E \times \bar{E}) = \sum_{\beta} \sum_{\text{ind}(\beta) = 1} \sum_{u \in H(p,q) \otimes R} \mu_u(E \times \bar{E}).$$

Why is $d^2 = 0$? When $\text{ind}(\beta) = 1$, $d(E \times \bar{E}) = \Pi H(p,3) \otimes H(3,q) \otimes R$.

We have $\lambda(T_u H(p,q) \otimes R) \cong \lambda(T_u H(p,3) \otimes R)$

\[ \cong o_q \otimes o_s \otimes o_s \otimes o_p \]

This gets even more messy when you look at the product structure...

"And they never mentioned signs again."