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Given $u \in \mathcal{M}(p,q)/\mathbb{R}$ rigid, $\lambda(T_u \mathcal{M}(p,q)/\mathbb{R}) \cong \underline{\mathbb{R}}$. (trivial bundle over a point)

After trivializing the $\mathbb{R}$-action, get

$$\mathbb{R} \cong \lambda(T_u \mathcal{M}(p,q)) \cong o_q \otimes o_p^\vee ,$$

so we get isomorphism $\mu_u : o_p \to o_q$, which induces

$|\mu_u|_{\mathbb{R}} : |o_p|_{\mathbb{R}} \to |o_q|_{\mathbb{R}}$. ← in (a), we say that $\mathrm{sgn}(u) = +1$ if μ_u orientation preserving, -1 otherwise.

So, for $[x] \in |o_p|_\Lambda$, define

$$\partial([x]) = \sum_q \sum_{\substack{\beta \\ \mathrm{ind}(\beta)=1}} \sum_{u \in \mathcal{M}(p,q)/\mathbb{R}} T^{\omega(u)} \cdot \mu_u([x]) \cdot (-1)^{|x|}$$

Why is $\partial^2 = 0$? When $\mathrm{ind}(\beta) = 2$, $\partial(\mathcal{M}(p,q)/\mathbb{R}) = \coprod \mathcal{M}(p,s)/\mathbb{R} \times \mathcal{M}(s,q)/\mathbb{R}$

We have

$$\begin{tikzcd}
\lambda(T\partial \mathcal{M}(p,q)/\mathbb{R}) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowright \text{ (analyse signs)}"] & \lambda(T\mathcal{M}(p,s)/\mathbb{R}) \otimes \lambda(T\mathcal{M}(s,q)/\mathbb{R}) \arrow[d] \\
o_q \otimes o_p^\vee \arrow[r, "\cong"] & o_q \otimes o_s^\vee \otimes o_s \otimes o_p
\end{tikzcd}$$

This gets even more messy when we look at the product structure...

"And they never mentioned signs again".

18/04/16 Actually, let's talk about orientations again: how does the (S)Pin structure factor in to various isos / signed counts?

$p \in L_0 \cap L_1 \rightsquigarrow o_p$ orientation line (real $\mathbb{Z}/2\mathbb{Z}$-graded 1D vector space)

$$\text{"}\det(D_p)$$

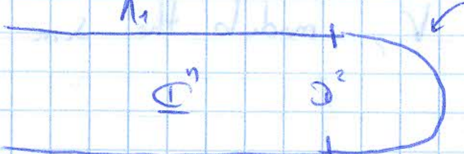
where D_p is a local Cauchy-Riemann operator

$T_pL_1 = \Lambda_1$

$F \subseteq \underline{\mathbb{C}^n}|_{\partial D^2}$ Lagr. subbundle

Λ_t path in $\Lambda(n)$.

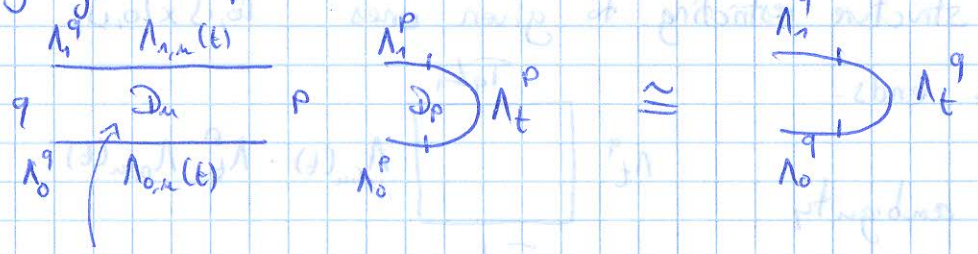
$\mathbb{H} \simeq D^2 \setminus \{-1\}$



← choose Λ_t, unique up to homotopy if L_0, L_1 graded, so Λ_0, Λ_1 lift to $\tilde{\Lambda}(n)$

$\Lambda_0 = T_pL_0$

Given p, q, $u \in \mathcal{M}(p,q)$, we wanted to say there is a gluing isomorphism

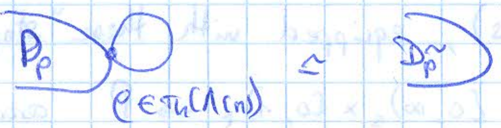


local CR operator associated to u on $u^*TX \simeq \mathbb{C}^n \to \mathbb{R}\times[0,1]$

inducing a map $\det(D_u)\otimes\det(D_p)\longrightarrow\det(D_q)$.

Such an iso depends on a choice of homotopy of paths in $\Lambda(n)$ rel endpoints between Λ_t^q and $\Lambda_{1,u}(t)\,\Lambda_t^p\cdot\Lambda_{0,u}(t)$.

Rem: they are always homotopic if L_0, L_1 are graded. Otherwise, factor in difference by an element of π_1, via



Actually, there is a non-trivial set of choices of homotopy, because

$$\pi_1(\underbrace{P_{a,b}\Lambda(n)}_{\simeq\,\Omega\Lambda(n)}) = \pi_2(\Lambda(n)) = \mathbb{Z}/2 \quad (n \text{ suff. large} \rightsquigarrow \text{stable range}).$$

The two choices lead to two different isos which differ by a sign.

Solution: equip each L_i with a Spin structure (or rel. Pin structure).

For each $p\in L_0\cap L_1$, make an extra choice: equip each Λ_t^p with a Spin structure restricting to given Spin structures already determined at endpoints

(want a Spin structure on $F \to [0,1]$ restricting to given one on endpoints).

Wanted: choose a homotopy $\Lambda_t^q \simeq \Lambda_{1,u}(t)\cdot\Lambda_t^p\cdot\Lambda_{0,u}(t)$. When viewed as bundles, both Λ_t^q and Λ_t^p now come with Spin structures $F_0\to[0,1]$, $F_1\to[0,1]$.