Given \( \nu \in H(p, q)/R \) rigid, \( \lambda(T_{\nu}H(p, q)/R) \cong R \) (trivial bundle over a point).

After trivializing the \( R \)-action, get

\[ R \cong \lambda(T_{\nu}H(p, q)/R) \cong O_{q} \otimes O_{p}, \]

so we get isomorphism \( \mu_{\nu} : O_{p} \to O_{q} \), which induces

\[ \mu_{\nu}^{-1} : O_{q} \to O_{p}. \]

So, for \( \text{for } (x, y) \in O_{p} \), define

\[ d(x, y) = \sum_{q} \sum_{p} \sum_{(x, y) \in H(p, q)/R} \lambda_{\nu}(x, y) \cdot \mu_{\nu}^{-1}(x, y) \cdot (-x, y). \]

Why is \( d^{2} = 0 \)? When \( \text{ind}(\nu) = 2 \),

\[ d^{2}(H(p, q)/R) = \prod H(p, q)/R \times H(p, q)/R \]

We have \( \lambda(T_{\nu}H(p, q)/R) \cong \lambda(T_{\nu}H(p, q)/R) \otimes \lambda(T_{\nu}H(p, q)/R) \).

This gets even more messy when we look at the product structure...

"And they never mentioned signs again!"

18/04/16

Actually, let's talk about orientations again: how does the \( (S) \text{Pin} \) structure factor in to various issues? signed counts?

\( p \in L_{0} \cup L_{1} \) as \( \nu \) orientation line (real \( \mathbb{Z}_{2} \)-graded 1D vector space)

\[ \text{"det}(D_{\nu}) \]

where \( D_{\nu} \) is a local Cauchy–Riemann operator.

\[ \tilde{H} = \mathbb{D} \setminus \{ -\} \]

\[ \tilde{\Lambda}_{1}, \tilde{\Lambda}_{1} \]

\[ \tilde{\Lambda}_{0} \]

\[ T_{p}L_{0} \]

\[ \Lambda_{0}, \Lambda_{1} \]

(i) \( L_{0}, L_{1} \) graded, so \( \Lambda_{0}, \Lambda_{1} \) lift

\[ \tilde{\Lambda}(n) \]
Given $p,q$ and $\mu \in K(p,q)$, we want to say there is a gluing isomorphism:

\[
\begin{array}{ccc}
\Lambda^q_{\mu} & \xrightarrow{\phi} & \Lambda^q_p \\
\downarrow \partial_\alpha & & \downarrow \partial_\alpha \\
\Lambda^p_{\alpha,\mu} & \xrightarrow{\phi} & \Lambda^p_\alpha \\
\end{array}
\]

The local CR operator associated to $\mu$ is $\mu^*T_X \cong e^n$

\[
\begin{array}{c}
\mu^*T_X \cong e^n \\
\end{array}
\]

 inducing a map $\det(D_w) \otimes \det(D_p) \rightarrow \det(D_q)$

Such an iso depends on a choice of homotopy of paths $\Lambda(n)_{\text{rel endpoints}}$ between $\Lambda^q$ and $\Lambda^q_{\alpha,\mu}$.

Rem.: they are always homotopic if $L_0, L_1$ are graded. Otherwise, factor in difference by an element of $\pi_0$, i.e.,

\[
\begin{array}{c}
\end{array}
\]

Actually, there is a non-trivial set of choices of homotopy, because

\[
\pi_1(P_{\alpha,\mu}, \Lambda(n)) \cong \pi_2(\Lambda(n)) = \mathbb{Z}/2 \quad \text{(in suff. large stable range)}
\]

The two choices lead to two different isos which differ by a sign.

Solution: equip each $L_i$ with a Spin structure (or rel. Pin structure). For each $p \in L_0 \cap L_1$, make an extra choice: equip each $\Lambda^p_\alpha$ with a Spin structure restricting to given Spin structures already determined at endpoints.

Wanted: choose a homotopy $\Lambda^q_\alpha \cong \Lambda^q_{\alpha,\mu}$ (i.e. $\Lambda^p_\alpha : \Lambda^p_{\alpha,\mu}$). When viewed as bundles, both $\Lambda^q_\alpha$ and $\Lambda^p_\alpha$ now come with a Spin structure $\mathbb{Z}/2 \times \mathbb{Z}/2$.
Now, there is a unique homotopy of paths rel end points, which, when thought of as a bundle, carries an $I$ from a Spin structure restricting to given ones $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ at various ends.

\[ \Lambda^3 \square \Lambda^1 \Lambda^0 = \Lambda^0 \Lambda^1 \Lambda^2 \]

Resolves ambiguity.

**Product structures:**

Say we have defined

\[
\begin{align*}
\text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2 ; \mathcal{H}_{0,0,0} \right) &= : \text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_1 \right) \\
\text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2 ; \mathcal{H}_{1,0,0} \right) &= : \text{CF}^* \left( \mathcal{L}_1, \mathcal{L}_2 \right) \\
\text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2 ; \mathcal{H}_{0,1,0} \right) &= : \text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_2 \right) \\
\text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2 ; \mathcal{H}_{0,0,1} \right) &= : \text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_1 \right)
\end{align*}
\]

Review the construction of a product map

\[ p^2 : \text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_2 \right) \otimes \text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_1 \right) \to \text{CF}^* \left( \mathcal{L}_0, \mathcal{L}_2 \right) \]

Consider $P = \mathbb{D}^2 \setminus \{3 \text{ boundary points}\}$, equipped with these "strip-like ends".

\[ \varepsilon^+ : [0, \infty) \times [0, 1] \to P \]

around \( z_1, z_2 \) or \( z_3 \).

\[ \varepsilon^- : (-\infty, 0] \times [0, 1] \to P \]

around $z_5$.

Equip $P$ with

1. A 1-form $\alpha$
2. Hamiltonian perturbation term $H_P : P \to C^0(\mathbb{R}, \mathbb{R})$
3. $P$-dependent almost complex structure
4. "Lagrangian labels" for $P$.
such that a consistency condition is satisfied near each end, as in the picture.

Rem: in some nice cases, we may be able to e.g. choose \( \phi = 0 \) or \( J_0 \) \( \phi \)-independent (\& integrable?) in order to make computations.

But it is not generally possible, e.g. with \( L_0 = L_1 = L_2 = L \).

Now, given \( z_2 \in X_{L_0, L_2} \) (time-1 chords \( L_0 \to L_2 \), "intersection points"), \( z_1 \in X_{L_1, L_2} \), consider

\[
H(z_1, z_2) = \begin{cases}
\mu : \mathcal{P} \to \mathcal{X} \\
\mu \big|_{\mathcal{P}_p} \in L_i \\
\lim_{s \to 0^+} (\mathcal{E}_0^s)^* \mu (s, b) = z_1 \\
\lim_{s \to -0^+} (\mathcal{E}_0^s)^* \mu (s, b) = z_2
\end{cases}
\]

Rems: strip-like ends help

(a) Analytically, in defining \( W^{k, p}(\mathcal{P}, \mathcal{X}, L_i) \), asymptotics with exponential decay

(b) in establishing in compactness analysis that when a strip breaks off, the equations solved is exactly Floer's equations for \( H_{L_1, L_2} \).

In nice cases (e.g., no disc or sphere bubbling), for generic \( \phi, J \):

1. \( H(z_0, z_2) \) is a manifold of dimension \( \deg(z_0) - \deg(z_1) - \deg(z_2) \)

2. \( \text{transversality argument} \ + \ \text{index calculation} \ + \ \text{gluing argument of local CR operators} \)

\[\text{ind}(D_0 \# z_0) = \text{ind}(D_0) + \text{ind}(D_{z_0})\]

3. Gromov's compactness + gluing: in the absence of bad bubbles, \( H \) is compactifiable with exactly 3 types of codim-1 boundary.
("the energy accumulates near punctures")

(i) \[ z_0 \to z_1 \to z_2 \]

(ii) \[ z_0 \to z_2 \] 
\[ \frac{z_2}{z_0} = z \]
\[ \frac{z_0}{z_2} = z \]
\[ \frac{z_0}{z_1} = z \]
\[ \frac{z_1}{z_2} = z \]
\[ \frac{z_1}{z_0} = z \]
\[ \frac{z_2}{z_1} = z \]
\[ \frac{z_2}{z_0} = z \]

The equations we solve in these broken off parts are \( J \)
\[ \partial L_1, L_2, \partial L_0, L_1, \partial L_0, L_2 \] thanks
to the consistency conditions.

(iii) \[ z_0 \to z_1 \to z_2 \]

(3) \( M \) is orientable "rel orientations at its ends", i.e.
there exists a canonical isomorphism (if we have fixed Spin structures, etc.)
\[ \chi(TM(z_0, z_1, z_2)) \cong O_{z_0} \otimes O_{z_1} \otimes O_{z_2} \]

With all of this, we can define
\[ p^2(z_2, z_1) = \sum_{\beta \in \beta^{(2, z_0, z_1, z_2)}} \chi_{\beta}(z_0, z_1, z_2, \beta) \cdot \frac{E(p)}{z_0^{2z_1}} \cdot (-1)^{deg \beta} \]

In a compact 0-dim manifold

Let see last lecture for how to turn this into a signed count.

Because the signed count of \( \partial (1) \) component of \( M(z_0, z_1, z_2) \) is 0, we get

**Proposition**

\[ p^2(z_2, z_1) = -p^2(p^2(z_2, z_1)) + \sum_{\beta \in \beta^{(2, z_0, z_1, z_2)}} \chi_{\beta}(z_0, z_1, z_2, \beta) \cdot \frac{E(p)}{z_0^{2z_1}} \cdot (-1)^{deg \beta} \]

\[ \partial z_2, \partial z_1, \partial z_0 \]

so \( p^2 \) is a chain map \( p^2 : C_2 \otimes D_{z_2} \to E \),

\[ \partial d = d_e \otimes id + id \otimes d_e \]

and so it depends on \( \partial z_0 \) but not on \( \partial z_1 \).

\[ \left[ \begin{array}{c} p^2(x_p, x_p, J_p) \end{array} \right] : HF^*(L_0, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2) \]

**Lemma** "homotopy of cho.co.s" \[ \left[ \begin{array}{c} p^2(x_p, x_p, J_p) \end{array} \right] \] is independent of the choices of \( \{ x_p, x_p, J_p \} \) (in absence of bad bubbling).
Theorem [Donaldson] \( L^0 \) gives the composition in a category, the Donaldson-Fukaya category \( \Phi_k^0(X) \) (= \( D^b_k(X) \), where the \( D \) stands for "Donaldson", not "derived")

**Objects.** Say, \( z_c(x) = 0 \), and fix a choice of fiberwise universal cover

\[ \Lambda \to \tilde{x} \]

or \( \mathbb{Z}/2 \) cover.

**Definition.** A Lagrangian brane is a triple \( L^\bullet = (L, \tilde{x}_L, \mathcal{P}) \)

where \( \mathcal{P} \) is a Span (or rel. \( D \)-) structure, and \( \tilde{x}_L \) is a grading

\[ \tilde{x}_L : \tilde{z}_L \to \mathbb{Z}/2 \]

or \( \mathbb{Z}/2 \) grading.

**Objects.** \( \text{Ob}(\Phi_k^0(X)) = \{ L^\bullet \} \)

\[ \text{Hom}(L^\bullet_0, L^\bullet) = \text{HF}^0(L^\bullet_0, L^\bullet) = \tilde{x}_L(\mathcal{C}^0(L^\bullet_0, L^\bullet), \tilde{x}_L(l_0) = a_{b_0}) \]

The composition \( \text{Hom}(L^\bullet, L^\bullet) \circ \text{Hom}(L^\bullet, L^\bullet) \to \text{Hom}(L^\bullet, L^\bullet) \)

is given by \( L^0 \circ (-, -) \).

**Need:** (a) identity morphisms \( [e_\tilde{x}] \in \text{Hom}(L, L), \forall \tilde{x} \), identity for \( L^0 \).

(b) composition is associative?

(a) Define \( e_L = \sum_{\gamma, \beta} \text{HF}(C_{\gamma, \beta}) \cdot \mathbb{E}_{(\gamma, \beta)} \cdot \mathbb{E}_{(\gamma, 0)} \cdot \mathbb{E}_{(\gamma, 0)} \)

where \( \gamma_L(\tilde{x}_{b_0}) = \left\{ \begin{array}{ll}
\mu : & \mathcal{H} = \mathcal{O}(\gamma) \\
\mathbb{C} \to \mathbb{C}
\end{array} \right. \)

We have \( \text{ind}(\mu) = -\deg(\tilde{x}) \). In particular, \( \gamma_L(\tilde{x}) \) is only rigid when \( \deg(\tilde{x}) = 0 \).
- Compactness + gluing \Rightarrow \text{codim-1 boundary} \quad \Rightarrow \quad \partial_{L,L}(e_L) = 0.

- Why a unit? E.g., \( (p^2) \left( (L^2), (e_L) \right) = \pm \alpha \).
  \( p^2(\alpha, e_L) \) counts
glue + homotopy
counting \( \text{index} \) a solution

Get an operator \( I : C^*F(K,L) \to \) counting the index 0 solutions to \( z^2 + \alpha = 0 \).

The only index 0 solutions are constant
\( \Rightarrow ~ \exists \alpha, I : 2 \text{Id}(\alpha) = \alpha \).

Counting solutions to the parametrized moduli space (by homotopy param.) gives a chain homotopy \( p^2(\alpha, e_L) - \alpha = \partial K + K D \).

Remark: if \( L \subset T^*L \) a section, then \( H = - \nabla^\alpha \), \( f : L \to \mathbb{R} \). If \( H \) is \( C^2 \)-small, then the associated \( e_L \) is \( \sum \rho \)
\( M(p) := \{ y : \mathbb{R} \to L \mid y = Df(x) \} = W^u(p) \).

In Morse theory, the product is

\[
\begin{pmatrix}
  \lambda & 0 \\
  0 & \rho
\end{pmatrix}
\]

Proposition (Fukaya-Oh) for \( L \subset T^*L \) a section, 3 canonical choices of grading & rel. Pin structures so that \( (HF^*(L), (p^2)) \cong H^*(L) \), as algebras with units.

The proof compares \( \xi_H \) to flowlines \( \gamma \) for \( "\text{no}" \) \( H \).