25/04/16 <u>So far</u>: (X^{2n}, ω) satisfying $(*)$

$K, L \subseteq X$ Lagrangians branes satisfying $(**)$ $\rightsquigarrow CF^*(K,L) \circlearrowleft \mu^1$ depending on $H_{K,L}, J_{K,L}$, but cohomologically independent of choices.

Given (L_0, L_1, L_2) we defined (depending on more choices, compatible with previous choices) a chain map

$$\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$$

st $[\mu^2]$ is independent of choices. This nearly gives a "categorical composition law" in $H^0 Fuk(X)$, <u>but</u> is it associative? On cohomology level yes, but not on the chain level.

For $L_0 \xrightarrow{x_{01}} L_1 \xrightarrow{x_{12}} L_2 \xrightarrow{x_{23}} L_3$, on the chain level, there is no reason for $\mu^2(x_{23}, \mu^2(x_{12}, x_{01}))$ to equal $\mu^2(\mu^2(x_{23}, x_{12}), x_{01})$:

counts counts

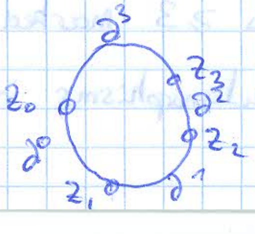
But we will show that the difference

$$\mu^2(x_{23}, \mu^2(x_{12}, x_{01})) - \mu^2(\mu^2(x_{23}, x_{12}), x_{01})$$

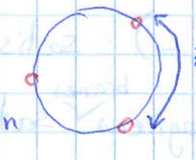
is null homotopic*, so $[\mu^2]$ is associative. * via a "geometric null homotopy"

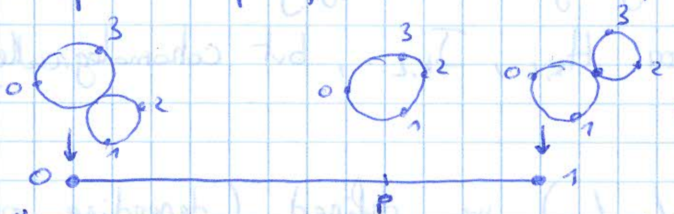
We will deduce this as a special case of constructions of "higher A_∞ homotopies".

Let $\mathcal{R}^3$ denote the space of discs with $3+1=4$ marked (inputs, output) points removed from the boundary, modulo automorphisms. A representative:



Up to biholomorphism, the position of 3 of the points (eg z_0, z_1, z_3) can be fixed at $-e^{2i\pi \frac{k}{3}}$.

So, $R^3 \underset{\text{diff}}{\cong} (0,1)$:  z_2 . There is a natural "Deligne Mumford" compactification of R^3 to $\overline{R^3} = [0,1]$:



(only 1 such picture, because $\underline{\geq}3$ pts on bubble.

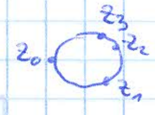
Notice that over 0 and 1, it looks like the pictures we drew for $\mu^2(-,\mu^2(-,-))$ and $\mu^2(\mu^2(-,-),-)$.

More generally, let R^d $(d \geq 2)$ be the space of discs with $d+1$ boundary marked points removed, labeled $\underset{\text{output}}{z_0}, \underbrace{z_1, \dots, z_d}_{\text{inputs}}$ cyclically ordered counterclockwise. On each $s \in R^d$, there is an induced decomposition $\partial S = \coprod_i \partial^i S$, where $\partial^i S$ is between z_i and z_{i+1} mod $d+1$.

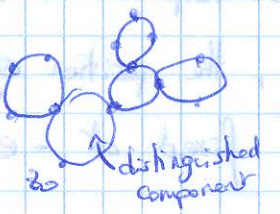
Up to biholomorphism, we can fix the positions of z_0, z_1, z_d.
$\Rightarrow \dim R^d = d-2$. Explicitly, $R^d = \mathrm{Conf}_{d+1}(\partial D^2) / \mathrm{Aut}(D^2)$.
There is a natural compactification to a $(d-2)$-dimensional polytope, the Stasheff associahedron $\overline{R^d}$.
* The top-dim boundary facets of $\overline{R^d}$ correspond to nodal degenerations

$\underset{\substack{\uparrow \\ \text{disks}}}{D} \rightsquigarrow \underset{\substack{\uparrow \\ \text{pair of disks, each carries} \geq 3 \\ \text{marked points, including nodes}}}{D_1 \cup D_2}$ 

There is a corresponding universal family of domains $\begin{matrix} \overline{S^d} \\ \downarrow \\ \overline{R^d} \end{matrix}$ $\underset{\substack{\uparrow \\ \text{repr. of Riemann surface}}}{}$; call S_r the fiber over $r \in \overline{R^d}$. ($\triangle$ the fiber is the point r, as top. space)

ex: $\overline{R^3} \cong [0,1]$, $S_p =$ , $S_0 =$ 

S_r is represented by a potentially nodal Riemann surface of the form



each component has ≥ 3 marked points (stability), mod automorphisms in each component.