So far: \((X^n_n, \omega)\) satisfying \((\star)\)

\[ K, L \in X \text{ Lagrangians} \text{ satisfying } (\star) \xrightarrow{\text{branes}} CF^*(K, L) \cong \mathbb{P}^n \]

depending on \(H_{KL}, S_{KL}\), but cohomologically independent of choices.

Given \((L_0, L_1, L_2)\) we defined (depending on more choices, compatible with previous choices) a chain map

\[ p^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2) \]

\(L[p^2]\) is independent of choices. This nearly gives a "categorical composition law" in \(H^*_{Fuk}(X)\), but is it associative? On cohomology level yes, but not on the chain level.

For \(L_0 \xrightarrow{x_{00}} L_1 \xrightarrow{x_{01}} L_2 \xrightarrow{x_{02}} L_3\), on the chain level, there is no reason for \(p^2(x_{02}, p(x_{01}, x_{01}))\) to equal \(p^2(p(x_{02}, x_{02}), x_{00})\).

But we will show that the difference

\[ p^2(x_{02}, p^2(x_{01}, x_{01})) - p^2(p^2(x_{02}, x_{02}), x_{00}) \]

is null homotopic, so \([p^2]\) is associative.

We will deduce this as a special case of constructions of "higher \(A_\infty\) homotopies."

Let \(R^3\) denote the space of discs with \(3+1=4\) marked points on the boundary, modulo automorphisms. A representative \(U_p\) to biholomorphism, the position of 3 of the points (e.g. \(z_0, z_1, z_2\)) can be fixed at \(-e\).
$S, \mathbb{R}^3 / \mathfrak{g}(0,1)$. There is a natural Deligne Mumford compactification of $\mathbb{R}^3$ to $\overline{\mathbb{R}^3} = \mathbb{R}^3 \cup \{0,1\}$.

Notice that over $0$ and $1$, it looks like the pictures we drew for $\mathbb{C}^\infty(-(\cdot,\cdot))$ and $\mathbb{C}^\infty(\cdot(\cdot,\cdot))$.

More generally, let $R^d(\mathbb{D}^2)$ be the space of discs with $d+1$ boundary marked points removed, labeled $z_0, z_1, \ldots, z_d$ cyclically ordered counterclockwise. On each $r \in R^d$, there is an induced decomposition $2R = \bigcup \partial S$, where $\partial S$ is between $z_i$ and $z_{i+1}$ mod $d + 1$.

Up to bilharmonic, we can fix the positions of $z_0, z_1, z_d$.

$\dim R^d = d - 2$. Explicitly, $R^d = \text{Conf}_{d+1}(\mathbb{D})/\text{Aut}(\mathbb{D})$.

There is a natural compactification to a $(d-2)$-dimensional polytope, the Stasheff associahedron $\overline{R^d}$.

* The top-dim boundary faces of $\overline{R^d}$ correspond to nodal degenerations $D = D_1 \cup \cdots \cup D_k$ of discs, each carries 3 marked points, including nodes.

There is a corresponding universal family of domains $\overline{R^d} \to \mathbb{C}^\infty(\cdot(\cdot,\cdot))$ in the fiber over $r \in \overline{R^d}$. (\cite{the fiber is the point $r$ of top space}) 

$S_r = \overline{R^3} \setminus \overline{2r}$, $S_0 = \{r_0, r_1, r_2\}$, $S_0 = \overline{\mathbb{C}^\infty(\cdot(\cdot,\cdot))}$.

$S_r$ is represented by a potentially nodal Riemann surface of the form each component has 3 marked points (stability), mod automorphisms in each component.
There is an underlying combinatorial type of a tree:
\[ \text{Codim (} \mathcal{R}_T^d \text{)} = \text{interior nodes of the tree.} \]

Now, we are going to put strip-like ends. Recall:
\[ Z_+ = [0,1] \times [0,\infty) \]
\[ Z_- = [0,1] \times (-\infty, 0] \]

**Definition:** A choice of strip-like ends is, for each \( i \in \{1, \ldots, d\} \) and for \( r \in R^d \), \( e^+_i : Z_+ \to S_r \) around \( z_i \), \( e^-_i : Z_- \to S_r \) around \( z_0 \), varying smoothly in \( r \).

If \( r \in R^d \), \( e^+_i : Z_+ \to S_r \) around each node, in a manner dictated by the picture:

![Diagram showing strip-like ends](image)

Given such a choice, note that we obtain a gluing map
\[ (\partial \mathcal{R}_T^d) \times [0,\infty) \to R^d \]
given by connect summing using the strip-like ends coordinates:

\[ \left( \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_d \\ \varepsilon \\ \lambda \end{array} \right) \to \left( \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_d \\ \varepsilon \\ \lambda \end{array} \right) \]

\( R^d \) is a manifold with corners.

**Definition:** Fix a consistent family of strip-like ends on \( R^d \), for all \( d \) (inductively). A universal and consistent choice of Floer data is a smoothly varying choice, inductively, for each \( d \) and each \( d \)-tuple \((L_0, \ldots, L_d)\) of objects ("Lagrangian branes"), for each \( s \in R^d \), of:
(a) Hamiltonian term \( H: S \to C^\infty(x; \mathbb{R}) \) with
\[
(E^*_\ell)^* H = H_{L_{\ell_1}, L_{\ell_2}}; \quad (E^*_0)^* H = H_{L_0, L_d}
\]
(b) Tame/compatible almost complex structure \( J: S \to \mathbb{R} \) with
\[
(E^*_\ell)^* J = J_{L_{\ell_1}, L_{\ell_2}}; \quad (E^*_0)^* J = J_{L_0, L_d}
\]
(c) 1-form \( \alpha \) on \( S \) with \( (E^*_\ell)^* \alpha = dt \).

This choice should be
* smoothly varying in \( S \)
* consistent, meaning that the restriction of the data to a corner \( S \) of a corner structure with induced suitable labelings \((L_{\ell_1}, L_{\ell_2})\) agrees with choices already made for this tuple \((L_{\ell_1}, L_{\ell_2})\) and \( S \in \mathbb{R}^{d_1} \), \( d_1 < d \), and that these choices vary smoothly across the corner charts.

\[
[S_1] \in S^2 \\
[S_2] \in S^3 \\
[S_3] \in S^4 \\
\rightarrow [S, (x_0, x_2, x_3)] \in S^9.
\]

So, the choices of (a), (b), (c) for internal vertices of the tree are determined by this consistency condition.

**Proposition.** Universal and consistent set of choices exist.

**Proof:** space of choices is contractible; this helps us to proceed inductively in defining consistent choices.

Make such a choice. Given a tuple \((L_0, \ldots, L_d)\) and
chords/intersection points \( x_i \in X_{L_{\ell_1}, L_{\ell_2}} \), \( x_0 \in X_{L_0, L_d} \), \( x_d \in \mathbb{R} \).

\[
R^d(x_0, x_1, \ldots, x_d).
\]
\[ \mathbb{R}^d(x_0, x_1, \ldots, x_n) = \begin{cases} r \in \mathbb{R}^d, \quad \mu : S_r \to X \\
(\xi_i, s) \in L_i \\
\ell_{m} \left( (\xi_i^\prime, m) \right) (s, \xi) = x_i \\
\ell_{m \cdot n} \left( (\xi_0^\prime, m) \right) (s, \xi) = x_0 \\
(d_n - X_{S_r} \otimes X_{S_r}) = 0 \\
\text{Hom of } \mathbb{R}^d_{X_r \cdot n} \\
\text{from } (\mathbb{R}^d_{X_r \cdot n}) \end{cases} \]

Remarks:

(i) Transversality and dimension: for generic choices, in the absence of "bad" bubbling, \( \mathbb{R}^d(x_0, x_1, \ldots, x_n) \) is a manifold of dimension \( d - 2 + \deg(x_0) - \sum_{i=1}^d \deg(x_i) \) for genus from \( \dim \mathbb{R}^d \) assuming \( \ell \) are graded.

(ii) In the degenerate case, \( d = 1 \), i.e., step, this agrees with \( \dim \left( \mathbb{R}^d(x_0, x_n) \right) \).

(iii) \( \mathbb{R}^d(x_0, x_1, \ldots, x_n) \) is canonically oriented rel ends and a choice of orientation of \( \mathbb{R}^{d - 2} \).

// \lambda(\mathcal{T} \mathbb{R}^d(x_0, x_1, \ldots, x_n)) = \lambda(\mathcal{T} \mathbb{R}^{d - 2}) \otimes o_{x_0} \otimes o_{x_1} \otimes \cdots \otimes o_{x_n} \).

(iii) Compactness + gluing: the limit configurations allowed by

- Gromov compactness
- Bubbling of spheres/discs, excluded by \((x), (x\#)\) at beginning.
- Bubbling of strips at marked points.
- Degeneration of domain to \( \partial \mathbb{R}^d \).

\[ \Rightarrow \text{Gromov-Floer compactification } \overline{\mathbb{R}^d(x_0, x_1, \ldots, x_n)} \text{, which in codim } 1 \]

is covered by the images of the natural inclusions of

\[ \mathbb{R}^d(x_0, \ldots) \times \mathbb{R}^d(x, \ldots) \]

\( d_n, d^2 \geq 1 \)

\( d_n + d_n - 1 = d \)
Define \( p^d : CF^*(L_d^-, L_d) \otimes \cdots \otimes CF^*(L_0^-, L_0) \to CF^*(L_0^-, L_0) \) :
\[
p^d(x_1, \ldots, x_n) = \sum_{x_{a+} \in X_{a+}, L_d} (-1)^{\deg(x_{a+}) + 2 - d} R^d(x_{a+}, x_{a+}, \ldots, x_n, \beta) \cdot \frac{E^d(\beta, x_{a+})}{x_{a+}}
\]
where \( R^d(x_0, x_1, x_2, \beta) \) is signed as before, and \( x_\alpha = \sum_i \epsilon \deg(x_i) \), and \( R^d(x_0, x_1, \ldots, x_n) = \frac{1}{\prod_{\beta \in \Pi_0^d(x_\alpha)} R^d(x_0, x_1, x_2, \beta) \cdot \prod_{i \neq j} \frac{1}{\deg x_{i-1} - \epsilon \deg x_{j+1}} \), with \( \alpha \) - equation.

**Proposition.** \( \sum \epsilon (-1)^i p^{d-k+i}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = 0. \)

**Proof.** By picture and compactness analysis.

First few equations:

- \( d = 1 \): \( p^0 \circ p^1 = 0 \), i.e. \( p^0 \) is a differential.
- \( d = 2 \): \( p^2(p^0 \circ p^1) = p^2(p^1 \circ p^0) + p^2(p^0 \circ p^0)(-1) = 0 \), i.e. \( p^2 \) is a chain map, so descends to cohomology with \( p^0 \).
- \( d = 3 \): \( p^3 p^2(x_0, x_1, x_2, x_3) = p^3(p^1(x_0, x_1, x_2) \otimes p^2(x_0, x_1, x_3)) \otimes p^1(x_0, x_2, p^0(x_1)) \)
  \[
  = p^2(x_0, p^0(x_1, x_2)) \otimes p^2(p^1(x_0, x_2), x_3).
  \]

The associator \( p^3 \) is a chain homotopy between this associator and 0, as desired.

- Higher homotopies...

**Note:** \( p^k \) \( (k > 2) \) don't descend to \( H^* \)!

**Theorem:** \( (\mathfrak{F}(k), p^0) \) is an invariant of \( X \) up to quasi-isomorphism (next time).