Let $A$ be a field.

**Definition:** a $(\mathbb{Z} \text{-graded})$ $A_{\infty}$-category (over $A$) $\mathcal{E}$ consists of the following data:

* a set of objects $\text{ob} \mathcal{E}$
* for any $k, l \in \text{ob} \mathcal{E}$, a $\mathbb{Z}$-graded vector space $\text{hom}_\mathcal{E}(k, l)$
* for $k > 1$ and any $(k_{+1})$-tuple $(k_0, \ldots, k_k)$ in $\text{ob} \mathcal{E}$, "higher composition map" $p^k_i : \text{hom}_\mathcal{E}(k_0, k_{+1}) \otimes \cdots \otimes \text{hom}_\mathcal{E}(k_k, k_0) \to \text{hom}_\mathcal{E}(k_0, k_k)$ of degree $2 - k$, satisfying $A_{\infty}$-relations: for each $d > 1$

$$\sum_{i=1}^{d+1} (-1)^i p^{d+1-i} \left( x_1, \ldots, x_{d+1}, p^i(x_{d-i+1}, \ldots, x_{d+1}), x_1, \ldots, x_d \right) = 0.$$

We showed:

**Theorem:** for $(x^a, \omega)$ symplectic satisfying $(\ast)$ (and maybe $\Delta_\omega(x) = 0$ and fixed $\tilde{\mathcal{G}} \to L \to X$), we can make choices to define an $A_{\infty}$ category over $A$ whose objects are Lagrangian branes $L = (L, \tilde{\mathcal{G}}, P)$ satisfying $(\ast \ast)$. We get called $\text{Fuk}_\omega (X)$, set of choices.
What choices?

* for each \((L_0, L_1)\), choose \(H_{L_0, L_1}, J_{L_0, L_1}\), to get
  \[ \text{hom}_F (L_0, L_1) := \text{CF}^* (L_0, L_1; H_{L_0, L_1}, J_{L_0, L_1}) \to \mathbb{P}^1 \]

* for each \((L_0, L_1, L_2)\), choose

\[ \begin{array}{c}
H_{L_0, L_1, L_2} \\
J_{L_0, L_1, L_2} \\
\delta_S
\end{array} \]

* for each \((L_0, \ldots, L_d)\), choose \(H_S, J_S, \alpha_S\) over \(S \in \mathbb{R}^d\), consistent universal choices.

Rem. \(A_\infty\) categories do not induce ordinary categories (i.e. no forgetful functor), before taking \(H^0\).

**Definition:** an \(A_\infty\)-functor \(F: \mathcal{E} \to \mathcal{D}\) between \(A_\infty\)-categories is the following data:

1. a map \(F: \text{ob} \mathcal{E} \to \text{ob} \mathcal{D}\)

2. for all \(d \geq 1\) and \((L_0, \ldots, L_d)\) in \(\text{ob} \mathcal{E}\), a linear map
   \[ F^d: \text{hom}_\mathcal{E} (L_d, L_0) \otimes \cdots \otimes \text{hom}_\mathcal{E} (L_0, L_0) \to \text{hom}_\mathcal{D} (F(L_0), F(L_d)) \]
   of degree \(d\) satisfying the \(A_\infty\)-functor relations: for each \(d\) and \(x_0, \ldots, x_n\) composable morphisms in \(\mathcal{E}\),
   \[ \sum (-1)^{i+j} F^{d-s-i} (x_{d-i}, \ldots, x_i, x_{i+1}, \ldots, x_n) \]
   \[ = \sum_{i, j, k} \mu^s_{d-i, j} (F^{i+1} (x_{d-i}), F^j (\ldots), \ldots, F^k (\ldots, x_i)) \]

Let us examine that:

* \(d=1\): \(F^0 p^0_{F^0} = p^0_F F^0\), i.e. \(F^0\) is a chain map.

* \(d=2\): \(F^2 p^2_F(x_0, x_n) = p^2_F(F^2_F(x_0), F(x_n)) \pm F^2_F(p^2_F(x_0), x_n) + F^2_F(x_n, p^2_F(x_0)) + p^2_F F^2_F(x_0, x_n)\),

i.e. \([F^0] p^0_{F^0}([x_0]_{F^0}, [x_n]_{F^0}) = [p^0_F] ([F^0_F([x_0]_{F^0}), [x_n]_{F^0}] + [F^2_F([x_0]_{F^0}, [x_n]_{F^0}) + [p^2_F] ([F^0_F([x_0]_{F^0}), [x_n]_{F^0}])].\)
Rem: $F : \mathcal{A} \to \mathcal{B}$ a functor between honest categories (except maybe they don't have a cot.).

A functor $F : \mathcal{E} \to \mathcal{D}$ is \textit{faithful} if $[F] : \mathcal{E} \to \mathcal{D}$ is essentially surjective, i.e., every object is isomorphic to an object in the image.

Two objects $x, y$ in a graded category $C$ are isomorphic if $f \in \text{Hom}^\bullet(x, y)$, $g \in \text{Hom}^\bullet(y, x)$ with $f \circ g = \text{id}_x$ and $g \circ f = \text{id}_y$.

\[ \textbf{Theorem:} \quad x \mapsto \text{Fuk}_s(x) \text{ is well-defined up to quasi-equivalence.} \]

Co get $\text{Fuk}(x)$, well-defined up to quasi-equivalence.

Rem: if $E$ has just one object $L$, the data of an $\mathcal{A}_\infty$-category reduces to a graded vector space $A := \text{Hom}^\bullet(L, L)$ and for any $d \geq 1$, $\mu_d : A^{[d]} \to A$ of degree $d - 1$ satisfying the $\mathcal{A}_\infty$-relations.

Why $\mathcal{A}_\infty$-algebras/categories?

Examples of $A$ as an $\mathcal{A}_\infty$-algebra with $\mu^0 = 0$, $\forall d > 2$, we get a differential graded algebra $(A, d = \mu^1)$.

Many examples: top space $\sim C^\bullet(Y)$ singular cochains is a DGA.

It is well understood in topology that the passage $C^\bullet(Y) \to H^\bullet(Y)$ (cochain algebra) loses lots of information.
ex: (Massey products) let $A$ be a DGA.

If $a, b, c$ are cocycles in $A$ with $[a, b] = [b, c] = 0$,
then we can define a homology class in $H^*(A)$, the Massey product of $a, b, c$, as follows:
- choose $\tau$ with $d\tau = a \cdot b$
- choose $\kappa$ with $d\kappa = b \cdot c$
- set $\langle a, b, c \rangle = \tau \cdot \kappa = a \cdot \kappa$

Check: $d (\text{this class}) = 0$ and cohomologically, the result is independent of choice (up to adding $[a], [c]$).

Point: "the triple product $[a, b, c]$ is zero for 2 different reasons", and the "sum of these reasons" is a secondary class.

There exists also higher Massey products for $n$-uples $(a_1, \ldots, a_n)$, with all $n-k$ Massey products of subsequences vanish for all $k > 1$.

ex: $B = \text{Borromean rings in } S^3$. We can check that as algebras,
$H^*(S^3 \setminus B) \cong H^*(S^3 \setminus 3 \text{ copies of unknot, unlinked}),$
but there are non-trivial Massey products on $H^*(S^3 \setminus B) \cong H_{d-1}(S^3, B)$
on $x_1, x_2, x_3$ in degree 1 Poincare dual to the standard bounding disks.

- in "higher linking".

We would like to retain that information. One use of $A_\infty$ structures:

[Theorem: ["Homological Perturbation lemma", "Transfer Theorem"]] Given a DGA $A$ (or $A_\infty$ algebra $A$), there is an $A_\infty$-structure on $H^*(A, d)$ (with $p^i = 0$), which is quasi-isomorphic to the original $A$.

Unfortunately, this $A_\infty$-structure on $B$ is not unique; for instance, given any $D$ with a sequence $F^*: D^{op} \to B$ with $F^*$ an isomorphism, we can pull back $F^*\mu_B$ to get an $A_\infty$-structure on $D$, and
\{F^{d}\} induces a quasi-isomorphism \((D, F^{*}p_{B}) \sim (B, p_{B})\)

\((B, p_{B})\) knows about all Massey products in \(A\)!

(Forget: a morphism \(F: C \rightarrow D\) is maps \(F^{d}: C^{\otimes d} \rightarrow D^{\otimes d}\), satisfying the functor equations).

Also, \(rk\ B\) is strictly smaller than \(rk\ A\).

Other advantages of \(A_{\infty}\) algebras:

Rational homotopy theory: says that the "rational homotopy type" of a space \(Y\) is determined "up to quasi-isomorphism of DGAs" by \((C^{\bullet}(Y; \mathbb{Q}), d, \cdot)\).

Problem: ordinarily, define a map of DGAs to be a chain map \(F: A \rightarrow B\) intertwining \(\cdot_{A}, \cdot_{B}\) strictly, and \(F\) is a quasi-iso if \([F]: H^{*}A \rightleftarrows H^{*}B\). But a quasi-isomorphism of DGAs is NOT an equivalence relation, so quasi-iso of DGAs are not invertible.

\[\text{Theorem:}\] \(X, Y\) have the same rational homotopy type if, roughly,

3 zig-zag of DGA quasi-iso:

\[C^{\bullet}(X; \mathbb{Q}) \rightarrow D_{1} \leftarrow D_{2} \leftarrow \cdots \leftarrow C^{\bullet}(Y; \mathbb{Q})\]

This problem disappears in the \(A_{\infty}\) setting:

\[\text{Theorem:}\] \(A_{\infty}\) quasi-isomorphisms are invertible (up to homotopy):

given \(F: A \rightleftarrows B\) \(A_{\infty}\)-quasi-iso, \(3\) \(G: B \rightleftarrows A\) with \(F \circ G \sim id_{A}\)

and \(G \circ F \sim id_{B}\)

(may require working over a field, or being very careful about working over projectives)
Proof:

\[ \begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
H^*A & \rightarrow & H^*B \\
\end{array} \]

\text{easy to invert by recursion, hence no differential.}

Rem: a morphism of \( A_\infty \)-algebras is \( \{F: A^{\infty} \rightarrow B\} \), get \( F = \oplus F^n: TA \rightarrow B \), where \( TA \) is the bar complex of \( A \).

**Definition:** \( A \) \( (A_\infty/DGA) \) is \textbf{formal} if

\[ A \simeq_{A_\infty \text{ quasi-iso}} (H^*A, d = 0, \omega = (-\mu^3), \mu^3 = 0) \]

\( (=) \) no Massey products

\( (\Rightarrow) \) coincides with the notion of DGA formality: a DGA \( B \) is formal if

\[ B \simeq \vdash (H^*B, -) \]

\( \text{with DGA quasi-iso.} \)

**Example:** [Deligne - Griffiths - Morgan - Sullivan] if \( X \) Kähler and \( \omega, \alpha \rightarrow \infty \), then \( \mathbb{C}^*(X, \omega) \) is formal

(Hodge theory constructs a 2 steps zig-zag)

**Other classical appearance of \( A_\infty \)-structures:**

Stasheff: \( \Omega Y \), with composition \( \Omega Y \times \Omega Y \rightarrow \Omega Y \) is an \( \text{"} A_\infty \text{"} \) space. So \( \mathbb{C}^*(\Omega Y) \) is an \( A_\infty \)-algebra

\[ \begin{array}{ccc}
& & c \\
& a & b \\
\frac{1}{14} & \frac{1}{14} & \frac{1}{14} \\
& & \vdash \end{array} \quad \begin{array}{ccc}
& & c \\
& a & b \\
\frac{1}{14} & \frac{1}{14} & \frac{1}{14} \\
& & \vdash \end{array} \]

\text{higher homotopy higher...}
Back to Floer theory:

\[ L = T^* L \text{ a section}; \text{ equip with canonical brane structure \textbf{if} L spin} \]

\[ \mathbb{L} ( CF^*(L,L) = \text{hom}_{T^*(H^1(T^*L), L)}(L,L), \mu^* ) \text{ \textbf{A}_\infty algebra}; \text{ well-defined up} \]

\[ \text{to quasi-isomorphism.} \]

\[ \text{Rem: can define over } \mathbb{C} \text{ or } \mathbb{Z} ! \text{ (Because exact, so } \alpha_1 : P_{n \cdot} \to \mathbb{R}, \text{ and not a cover)}, \text{ (weigh all discs by } n). \]

\[ \text{Exactness tells us that the cactus is still finite, because the action of} \]

\[ \text{asymptotic chords gives a priori energy bounds.} \]

\[ \textbf{Theorem (Fukaya-Oh rationaliy, Abo aid over } \mathbb{Z}) \text{ \textbf{A}_\infty equivalence} \]

\[ CF^*(L,L) = C^*(L), \text{ so } CF^*(L,L) \text{ is formal iff } L \text{ is.} \]

Examples in Riemann surfaces:

\[ X_1 = \mathbb{T} / \langle \alpha_1 \rangle \]

\[ X_2 = \mathbb{Z} / \langle \alpha_1 \rangle \]

\[ \textbf{Theorem (Levitt-Prasad)} \text{ the Fukaya} \]

\[ \text{category with objects } (L_1, L_2) \text{ is} \]

\[ \text{not formal.} \]

\[ \text{In either cases, no visible discs, but "small discs" appear when we perturb.} \]

\[ \textbf{Pinchline: even these Fukaya categories are not combinatorially} \]

\[ \text{computable to all orders.} \]