02/05/16

Today: miscellanous topics.

1) Units in A_∞-categories / the Fukaya category

2) Local systems on Lagrangians

3) Cases in which L bounds discs, X has spheres, but things still work simply (monotone).

Units.

There are several possible notions of "units" / "identity morphisms" in an A_∞-category $(\mathcal{C}, \mu^\cdot)$.

Terminology: $\mathrm{Hom}^*_{\mathcal{C}}(A,B) := H^* \hom_{\mathcal{C}}(A,B)$.

Definition: an A_∞-category $\mathcal{C}$ is cohomologically unital ("c-unital", "h-unital") if, for every $X \in ob(\mathcal{C})$, $\exists\ [e_X] \in \mathrm{Hom}^0_{\mathcal{C}}(X,X)$ satisfying

$$\left[\mu^2(e_X, \sigma)\right] = (-1)^{\deg(\cdot)}[\mu^2(\sigma, e_Y)] = [\sigma], \quad [\sigma] \in \mathrm{Hom}^*(Y,X).$$

Let (X^{2n}, ω), satisfying $(*)$.

Proposition: $\mathcal{F}uk(X)$ is c-unital.

Proof: already given, when talking about Donaldson Fukaya category. □

There is a stricter notion which is frequently convenient.

Definition: an A_∞-category $\mathcal{C}$ is strictly unital if $\forall X \in ob(\mathcal{C})$, $\exists$ morphisms $e_X^+ \in \hom^0(X,X)$ with

* $\mu^2(e_X^+, \sigma) = (-1)^{\deg(\cdot)} \mu^2(\sigma, e_Y^+) = \sigma, \quad \forall \sigma \in \hom^\bullet(Y,X)$.
* $\mu^k(\ldots, e_X^+, \ldots) = 0$ for $k > 2$. $\quad \cdot\ \mu^1(e_X^+)$.

A priori, the geometric elements that we have produced $[e_L] \in HF^0(L,L)$ are not strict units. We need 2 choices for producing strict units:

- Use $[e_X]$ + higher compatibilities/homotopies to produce a homotopy unit ("keeps track of all homotopies" [Fukaya, FOOO]

$\rightsquigarrow$ strictly unital category $\mathcal{F}_h(x) \overset{q.i.}{\simeq} \mathcal{F}(x)$

$\uparrow$ Fukaya cat. with homotopy units.

- **Proposition** [Lefevre, Seidel] any c-unital A_∞-category $\mathcal{E}$ is quasi-iso to $\tilde{\mathcal{E}}$ strictly unital; we can assume $\tilde{\mathcal{E}}$ is minimal (ie $\mu^1 = 0$).

So, up to quasi-iso,

c-unital $\Leftrightarrow$ strictly unital $\Leftrightarrow$ homotopy unital.

Local systems:

There is an enlargement of the Fukaya category whose objects are Lagrangian branes equipped with local systems with flat connections [Kontsevich].

Given $\begin{matrix}\mathcal{E}_K \\ \downarrow \\ K\end{matrix}$, $\begin{matrix}\mathcal{E}_L \\ \downarrow \\ L\end{matrix}$ $\mathbb{R}$-local systems with flat connections, we can define

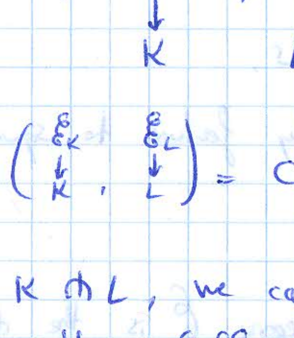
$$\hom_{\mathcal{F}}\left(\begin{matrix}\mathcal{E}_K \\ \downarrow \\ K\end{matrix}, \begin{matrix}\mathcal{E}_L \\ \downarrow \\ L\end{matrix}\right) = CF^\circ\left(\begin{matrix}\mathcal{E}_K \\ \downarrow \\ K\end{matrix}, \begin{matrix}\mathcal{E}_L \\ \downarrow \\ L\end{matrix}\right) = \bigoplus_{\substack{x \in \mathcal{X}_{K,L} \\ \text{time-1 chord}}} \underbrace{\mathrm{Hom}_{\mathbb{R}}\left((\mathcal{E}_K)_{x(1)}, (\mathcal{E}_L)_{x(0)}\right)}_{\text{"}x\text{-component"}}$$

If $K \pitchfork L$, we can chose $H_{K,L} = 0$, the x-component is just $\mathrm{Hom}_{\mathbb{R}}\left((\mathcal{E}_K)_p, (\mathcal{E}_L)_p\right)$

Given $y \in \mathcal{X}_{K,L}$ and $\Phi_x \in \hom_{\mathbb{R}}\left((\mathcal{E}_K)_{x(1)}, (\mathcal{E}_L)_{x(0)}\right)$, $\mu^1(\Phi_x)$ has "y-component"

$$:= \sum_{\substack{\beta, u \in \mathcal{M}(x,y,\beta)/\mathbb{R} \\ \mathrm{ind}(\beta)=1}} T^{E(\beta)} \cdot \mathrm{sgn}(u) \cdot \mathrm{hol}_{[\partial u]}(\Phi_x), \quad \text{where}$$

$$\mathrm{hol}_{[\partial u]}(\Phi_x) := \mathrm{hol}_{\gamma_0} \circ \Phi_x \circ \mathrm{hol}_{\gamma_1} \in \mathrm{Hom}\left((\mathcal{E}_K)_{y(0)}, (\mathcal{E}_L)_{y(1)}\right).$$



The earlier cases embed into this more general situation by $\begin{matrix}\mathbb{R} \\ \downarrow \\ L\end{matrix}$.