And there is a fully faithful embedding $\mathcal{E} \hookrightarrow \mathcal{E}^{\Delta} \hookrightarrow \mathcal{E}^{\Delta,\pi}$
idempotent pre-Δ hull.

11/05/16

Definition: an A_∞-category $\mathcal{E}$ is <u>pretriangulated</u> if

(a) • every closed morphism in $\mathcal{E}$ extends to an exact triangle

(b) • $\mathcal{E}$ is "closed under shifts": for every object X in $\mathcal{E}$, $\exists\, X[1]$ st

$\forall K$, $\begin{cases} hom^{\cdot}_{\mathcal{E}}(X, K[1]) = hom^{\cdot}(X,K)[1] \\ hom^{\cdot}_{\mathcal{E}}(X[1], K)[1] = hom^{\cdot}_{\mathcal{E}}(X,K) \end{cases}$ equality of chain complexes, compatible with A_∞-structures ⚠ signs

Rem: if (a), we can already define $X[1] := \text{Cone}(X \xrightarrow{0} 0)$. Note that $0 = \text{cone}(\mathbb{k} \xrightarrow{id} \mathbb{k})$ is in the category

⊛ (We always take $\mathcal{E}$ hom-unital).

But we also want $X[-1]$ to be define, hence (b).

Rem: $\mathcal{E}$ pretriangulated $\rightsquigarrow$ $H^0\mathcal{E}$ triangulated in usual sense.

Proposition: any (hom-unital) A_∞-category $\mathcal{E}$ has a (essentially unique) pretriangulated hull, ie $\hat{\mathcal{E}}$ pretriangulated with $\mathcal{E} \to \hat{\mathcal{E}}$ such that if $\mathcal{D}$ pretriangulated with $\mathcal{E} \hookrightarrow \mathcal{D}$, then $\hat{\mathcal{E}} \hookrightarrow \mathcal{D}$.
In fact, the restriction $fun(\hat{\mathcal{E}}, \mathcal{D}) \longrightarrow fun(\mathcal{E}, \mathcal{D})$ is a quasi-equivalence.

<u>Two methods of constructing this:</u>

1) <u>A_∞-modules over $\mathcal{E}$</u>: construct a pretriangulated category $Mod(\mathcal{E})$ with $\mathcal{E} \hookrightarrow Mod(\mathcal{E})$, and take the closure of $im(\mathcal{E})$ under finitely many cones and shifts.

2) <u>Twisted complexes</u> $Tw\,\mathcal{E}$: construct directly from $\mathcal{E}$ an enlargment whose objects are "complex of objects"

ex. $X_0 \xrightarrow{c_{01}} X_1 \xrightarrow{c_{12}} X_2$, $\mu^2(c_{12}, c_{01}) = \mu^1(c_{02})$: comp. is null homotopic instead of 0.

(Bondal - Kapranov, Kontsevich).

Let $\mathbb{k}$ be the ground field for $\mathcal{E}$. We will talk more about $\mathcal{U}$.

Recall that given $\mathbb{k}$, we have a DG category $\mathrm{Chain}_{\mathbb{k}}$ ($\mathrm{Ch}_{\mathbb{k}}$):

* objects $C^\bullet \circlearrowright d_{C^\bullet}$ ($C^\bullet$ graded vs, $\deg(d_{C^\bullet}) = +1$) chain complex
* morphisms $\mathrm{Mor}(C_1, C_2) = \hom_{\mathrm{Vect}}(C_1, C_2)$, with d, inherits grading from C_1 and C_2. This differential is

(§) $d(F) := F \circ d_{C_1} - d_{C_2} \circ F$ ($\pm$ signs depending on $\deg F$), so closed morphisms are chain maps.

Fact: given A_∞-categories $\mathcal{E}$ and $\mathcal{D}$, we get $nu\text{-}\mathrm{fun}(\mathcal{E}, \mathcal{D})$, category of (necessarily non-unital) A_∞-functors from $\mathcal{E}$ to $\mathcal{D}$. It is DG if $\mathcal{D}$ is.

Definition: the category $\mathrm{Mod}(\mathcal{E}) = \mathrm{Mod}\text{-}\mathcal{E}$ of (right) A_∞-~~functors~~ modules over $\mathcal{E}$ is $nu\text{-}\mathrm{fun}(\mathcal{E}^{op}, \mathrm{Ch}_{\mathbb{k}})$. In fact, it is a DG category.

Let's explicitely spell this out.

Definition: a (right) A_∞-module P over $\mathcal{E}$ is the data of:

- for every $X \in \mathrm{ob}\,\mathcal{E}$, a chain complex $P(X) \circlearrowright \mu_P^{1|0}$, degree 1
- for every $(d+1)$-tuple of objects $X_0, \dots, X_d$, "multiplication maps"
  $\mu_P^{1|d} : P(X_d) \otimes \hom_{\mathcal{E}}(X_{d-1}, X_d) \otimes \dots \otimes \hom(X_0, X_1) \to P(X_0)$ of degree $1-d$.

  Equivalently, $P^d : \hom(X_{d-1}, X_d) \otimes \dots \otimes \hom(X_0, X_1) \to \hom_{\mathrm{Ch}_{\mathbb{k}}}(P(X_d), P(X_0))$.

satisfying the A_∞-module equations (direct consequence of P A_∞-functor): for any k,

$$(*)\quad \sum_{i,j} \pm \mu_P^{1|k-j+1}(\underline{m}, x_k, \dots, x_{i+j+1}, \mu_{\mathcal{E}}^j(x_{i+j}, \dots, x_{i+1}), x_i, \dots, x_1) \quad (\mu_P^{1|0} \text{ allowed})$$
$$= \sum \pm \mu_P^{1|k-i}(\mu_P^{1|i}(\underline{m}, x_k, \dots, x_{k-i+1}), x_{k-i}, \dots, x_1)$$

First few: * $(\mu_P^{1|0})^2(\underline{m}) = 0$

* $\pm \mu_P^{1|0}(\mu_P^{1|1}(\underline{m}, x)) = \pm \mu_P^{1|1}(\mu_P^{1|0}(\underline{m}), x) \pm \mu_P^{1|1}(\underline{m}, \mu_{\mathcal{E}}^1(x))$,

  ie $\mu_P^{1|1}$ descends to cohomology: $H^\bullet(P(X)) \otimes \mathrm{Hom}(Y, X) \to H^\bullet(P(Y))$.