And there is a fully faithful embedding $\mathcal{E} \hookrightarrow \mathcal{E}_D \hookrightarrow \mathcal{E}_{h^0}$.

**Definition:** An $A_\infty$-category $\mathcal{E}$ is pretriangulated if

1. every closed morphism in $\mathcal{E}$ extends to an exact triangle
2. $\mathcal{E}$ is "closed under shifts": for every object $X$ in $\mathcal{E}$, $3X \in \mathcal{E}$
3. $\forall K, \forall \{X_n \in \mathcal{E} \mid n \geq 0\}$ equality of chain complexes, compatible with $A_\infty$-structures

**Rem.** if (a), we can already define $X(-1) = \text{Core } (X \rightarrow 0)$. Note that $0 = \text{core } (\text{id}_K \rightarrow K)$ is in the category.

**Rem.** (We always take $\mathcal{E}$ hom-unital).

But we also want $X[-1]$ to be defined, hence (b).

**Rem.** $\mathcal{E}$ pretriangulated $\Rightarrow$ $h^0\mathcal{E}$ triangulated in usual sense.

**Proposition:** any (hom-unital) $A_\infty$-category $\mathcal{E}$ has a (essentially unique) pretriangulated hull, i.e. $\mathcal{E}$ pretriangulated with $\mathcal{E} \hookrightarrow \mathcal{E}'$ such that if $\mathcal{D}$ pretriangulated with $\mathcal{E} \leftrightarrow \mathcal{D}$ then $\mathcal{E} \leftrightarrow \mathcal{D}$.

In fact, the restriction $\text{Fun}(\mathcal{E}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{E}, \mathcal{D})$ is a quasi-equivalence.

Two methods of constructing this:

1. $A_\infty$-modules over $\mathcal{E}$: construct a pretriangulated category $\text{Mod}(\mathcal{E})$ with $\mathcal{E} \leftrightarrow \text{Mod}(\mathcal{E})$, and take the closure of $\text{im} \mathcal{E}$ under finitely many cones and shifts.

2. Twisted complexes: Tw $\mathcal{E}$ - construct directly from $\mathcal{E}$ an enlargement whose objects are "complex of objects" $C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots$ with morphisms $p^k(C_n, C_{n+1}): p^k(C_{n+1}, C_n) = \text{comp} \circ 0$ and homotopy instead of 0.

(Bondal - Kapranov, Kontsevich)
Let $K$ be the ground field for $E$. We will talk more about $d$.

Recall that given $K$, we have a DG category $\text{Ch}_{K}^\bullet (\text{Ch}^\bullet_K)$. * objects: $C^* \otimes_K C$ (graded vs. $d(C^*):= r$) chain complex
* morphisms $\text{Mor}(C^*, C^\prime) := \text{Hom}_{\text{vert}}(C^*, C^\prime)$, with $d$, inherits grading from $C$ and $C^\prime$. This differential is $d := F \circ d_C - d_C \circ F$ ($\pm$ signs depending on $dC$), so closed morphisms are chain maps.

Fact: given $A_{\infty}$-categories $E$ and $D$, we get $nu$-functor $\text{nu-fun}(E, D)$, category of (non-unique) $A_{\infty}$-functors from $E$ to $D$. It's DG if $D$ is.

**Definition**: the category $\text{Mod}(E) := \text{Mod}_K$ of (right) $A_{\infty}$-functors modules over $E$ is $\text{nu-fun}(E^\text{op}, \text{Ch}_K)$. In fact, it is a DG category.

Let's explicitly spell this out.

**Definition**: a (right) $A_{\infty}$-module $P$ over $E$ is the data of:

* for every $x \in \text{ob} E$, a chain complex $P(x) \in \text{Ch}_{M}^\bullet$
* for every $(d+1)$-tuple of objects $X_0, \ldots, X_d$, "multiplication maps" $\mu^d_{P}: P(X_0) \otimes \text{Hom}_{E}(X_1, X_0) \otimes \cdots \otimes \text{Hom}_{E}(X_{d}, X_0) \to P(X_0)$ of degree $1 - d$

Equivalently, $P^d : \text{Hom}_{E}(X_1, X_0) \otimes \cdots \otimes \text{Hom}_{E}(X_d, X_0) \to \text{Hom}_{E}(P(X_d), P(X_0))$, satisfying the $A_{\infty}$-module equations (direct consequence of $P$ $A_{\infty}$-functor): for any $k$,

\[ \sum_{i,j} \mu^i_{P}(m, x_k, \ldots, x_{i+j+1}, \mu^j_{P}(x_{i+j}, \ldots, x_{d})) \]

(\emph{\mu^b_{P} allowed})

\[ \sum_{i,j} \pm \mu^i_{P}(m, x_k, \ldots, x_{i+j+1}, x_k, \ldots, x_d) \]

First few:

(\emph{\mu^b_{P} allowed})

\[ \pm \mu^0_{P}(m, x) \]

\[ \pm \mu^1_{P}(m, x) \]

\[ \pm \mu^2_{P}(m, x) \]

\[ \pm \mu^3_{P}(m, x) \]

ie $\mu^b_{P}$ descends to cohomology: $H^0(P(x)) \otimes \text{Hom}_{E}(Y, x) \to H^0(P(Y))$. 

Rem: when $E$ has one object $x$, $A := \text{hom}_E(x,x)$ a $A_\infty$-algebra.

A module over $E$ $\Rightarrow$ "$A_\infty$-module over $A$" a graded vector space $M := P(x)$, with maps $\pi^d : M \otimes A^d \rightarrow M$ of degree $d$ $(d > 0)$ satisfying $(\ast)$.

Example: let $K$ be any object in $E$.

2. $Y^r_K := \text{Yoneda module over } E := \text{hom}_E(-,K)$.

And $P^a_{Y^r_K} : Y^r_K(x_d) \otimes \text{hom}(x_{d-1},x_d) \otimes \ldots \otimes \text{hom}(x_0,x_1) \rightarrow Y^r_K(x_0)$.

If $E$ has only one object $X$, then $A := E \text{ad}(X)$, and we get $A$

as an $A_\infty$-module over itself.

What are morphisms in $\text{Mod}(E)$?

(Answer: $\text{Mod}(P_0,P_n) := $ space of $A_\infty$-pre-module morphisms $E \rightarrow \mathcal{A}$)

A pre-morphism from $P_0$ to $P_n$ is the data of

- a linear map $F^{\text{lin}} : P_0(x) \rightarrow P_n(x)$
- higher maps $F^{\text{cl}} : P_0(x_d) \otimes \text{hom}(x_{d-1},x_d) \otimes \ldots \otimes \text{hom}(x_0,x_1) \rightarrow P_0(x_0)$.

All together,

$\text{Mod}(E)(P_0,P_n) := \prod_{x_0,x_d \in \text{ob} E} \text{hom}_E(P_0(x_0),\text{hom}(x_0,x_1) \otimes \ldots \otimes \text{hom}(x_0,x_1),P_n(x_0))$.

Short-hand: (abbreviated from algebraic case)

$P_0 \otimes E^\text{ad} = \bigoplus_{x_0,x_d \in \text{ob} E} P_0(x_d) \otimes \text{hom}(x_{d-1},x_d) \otimes \ldots \otimes \text{hom}(x_0,x_1)$

and $P_0 \otimes TE := \bigoplus_{d \geq 0} P_0 \otimes E^\text{ad}$

Point: these $F^{\text{cl}}$ do not satisfy any equations. Just like in $E$,

Given a map $F = \oplus F^{\text{cl}} : P_0 \otimes TE \rightarrow P_n$ as before, there is a

natural extension $\hat{F} : P_0 \otimes \hat{E} \rightarrow P_n \otimes \hat{E}$,

$\hat{F}(p,x_0 \ldots ,x_n) = \sum_{d>0} F(p,x_0 \ldots ,x_{d-1}) \otimes x_d \ldots \otimes x_n$.  

\( \text{(x_0, \ldots, x_n)} \)
Similarly, given a $A_{\infty}$-structure $p : T_E \rightarrow E$, get a canonical
$p : T_E \rightarrow T_E \hat{p}(x_1, \ldots, x_n) = \sum \pm x_k \Theta \Theta \Theta p(x_{i_1}, \ldots, x_{i_r}) \Theta x_1 \Theta \Theta \Theta x_n$.

**Composition** given $G \in \text{hom}_{A_\infty}(E) (P_0, P_1) = \text{hom}_{\text{vec}}(P_0 \otimes T_E, P_1)$
$F \in \text{hom}_{A_\infty}(E) (P_0, P_1)$,
define $G \circ F := G \circ \hat{F}$, namely
$G \circ F(m, x_1, \ldots, x_n) = \sum \pm G(F(m, x_1, \ldots, x_n), x_1, \ldots, x_n)$.
and $S(F) := p_{P_0} \circ \hat{F} \circ F_0 \circ \hat{p}_{P_1} \circ \hat{F}(m, \hat{F} (\ldots))$.

It is a DG category.

The association $K \mapsto Y^0_K$ extends to an $A_\infty$-functor $Y : E \mapsto \text{Mod}(E)$
$K \mapsto Y^0_K : \text{hom}_{E}(K, L) \mapsto \text{hom}_{\text{Mod}(E)}(Y_K, Y_L)$. Part of this data is, for $K, L, x$,
map $Y^0_K(x) \rightarrow Y^0_L(x)$ for $\phi : \text{hom}_{E}(K, L)$.

**Proposition (A_\infty-Yoneda embedding)** if $E$ is hom-complete, $Y_K$ is coherent
full and faithful.

**Corollary**: any $A_\infty$-category is quasi-equivalent to a DG category.

*Note*: $	ext{Mod}(E)$ inherits the following operations from $\text{Ch}_{(k)}$:

(i) can take direct sum of modules $P_0, P_1 \mapsto P_0 \oplus P_1(x) := P_0(x) \oplus P_1(x)$.

(ii) can tensor with a fixed chain complex or vector space:
$V$ vs. $P$ module $\mapsto (V \otimes P)(x) = V \otimes (P(x))$.

(iii) can shift objects $P[\ell_0](x) := P(x + \ell)$.

(iv) mapping cone: recall that given a closed morphism $f : K \rightarrow L$ in $\text{Ch}_{(k)}$,
get $\text{Cone}(f) \in \text{ob}(\text{Ch}_{(k)})$
$K \oplus L$, with $d_{\text{Cone}(f)} = \begin{pmatrix} d_K & f \\ 0 & d_L \end{pmatrix}$,
filling into a triangle \[ K^0 \xrightarrow{f} L^i \]
\[ \xrightarrow{p} \text{Core}(f) \]

Rem: 3 non trivial Massey product between \( p, i, f \), namely:

\[ p \circ i = 0 \text{ on chain level, but } \circ f = S_{\text{Core}(f)}(\alpha) \]

and Massey \( (p, i, f) : p \circ \alpha = id_k \)

Similarly, given \( P_0, P_1 \text{ and } T : P_0 \to P_1 \) closed morphism, have \( \text{Core} \text{(F)} = P_0 \oplus P_1 \text{ with } \mu_{\text{Core}(f)} \text{ independent } \)

Idempotents: in \( \text{Ch}_{k} \), given \( p \in \text{hom}_{\text{Ch}_{k}}(C, C) \text{ with } p^2 = p \), we can split off \( C = \text{im}(p) \oplus \ldots \)

A chain map: \( p \circ p = p \circ p = 0 \) as \( p \text{ closed} \)

Similarly, given a \( A_{\text{mod}} \text{-module } P \text{ with "idempotent up to homotopy" } \)

\( \Leftrightarrow \) a morphism \( (p) \in H^0 \text{ hom}_{\text{Rad}(E)}(P, P) \text{ with } (p)^2 = (p) \)

we can split \( P = \text{im}(2p) \oplus \ldots \)

Definition: the **triangulated hull** of \( E \), denoted \( \hat{E} \) (or \( \text{Tw} \hat{E} \)), is the closure of \( Y_{k} (E) \subseteq \text{Mod}(E) \) under finitely many shifts and mapping cones.

The **split-closed triangulated hull** of \( E \), denoted \( \text{Tw}^* (E) \) or \( \text{Perf}(E) \), is the closure of \( Y(E) \) in \( \text{Rad}(E) \) under shifts, mapping cones and idempotent summands.

Both come equipped with embeddings (coh. full and faithful functors) \( E \subseteq \hat{E} \subseteq \text{Perf}(E) \)
If \( A \leq E \) a subcategory of \( E \), get \( \hat{A} \leq \hat{E} \) and \( \text{Perf}(A) \leq \text{Perf}(E) \).

**Definition:**
\[
\hat{D}(E) := H^0(\text{Tw } E) \quad \text{idempotent Rees-\-di} \text{ -completed derived category}
\]
\[
\hat{D}(E) := H^0(\text{Tw } E) \quad \text{derived category}
\]

**Rem.:** \( H^0(\text{Tw } E) \neq \text{Tw } H^0 E \) (which is useless)

**Rem.:** \( A = kE(x) \) and \( B = H_{\text{ext}}(kE(x)) \) have same category of perfect modules.

**Definition:** \( A \) **generates** \( E \) if \( \text{Tw } A \sim \text{Tw } E \) is a quasi-equivalence.

\( a \) in \( \text{Tw } E \), every object of \( E \) is \( \leq \) to an iterated mapping core of objects in \( A \).

**Nice fact:** in many cases, \( \text{Fuk}(x) \) is split-generated by a subcategory with finitely many Lagrangians. So, \( \text{Fuk}(x) \subseteq A\text{-mod} \) (or \( \text{perf}(A) \)) for \( A \) some \( A_n \)-algebra.

**Why "split-generation"?**
**Philosophy:** summand of objects = connected components of Lagrangians.

\[
\begin{array}{c}
\text{tw} \: E \\
\text{tw} \: A \\
\text{tw} \: \hat{E} \\
\text{tw} \: \hat{A} \\
\end{array}
\]

\( L_0, L_\alpha, L_\beta \) **"split-generate"** \( L_a \).