Draw \( \gamma \): for every \( L \in \mathbb{Z} \), get a generalized thimble \( \Delta^L \).
Note that it seems that

\[
\hom \left( \Delta^L, \Delta^K \right) : = \text{HF}^* \left( \phi_e \Delta^L, \Delta^K \right) \cong \text{HF}^* (L, K)
\]

**Theorem (Abavaid-Auroux-Katzarkov)**: This is true (with signs) (for lagrangian
branes when \( Y(Z) \) (normal bundle) is spin.

**Proposition (GAGA)**: There is a fully faithful \( A_0 \)-embedding

\[
F(Z) \subset \to F(E, W)
\]

**Theorem**: \( \Delta^{(e)} \) (any split-generating collection in \( E \) in sense of satisfying

"Abavaid-Auroux-Katzarkov's generation criterion")

- strictly stronger than split-generation.

Next time,

\[
\phi_{37} \quad C \quad F(E, W) \xrightarrow{\gamma} \quad F(M) \xrightarrow{\nu} \text{monodromy map}
\]

\[
\text{general fiber}
\]

23/05/16: Last time: \( (E, W) \) symplectic LG-model, e.g. a symplectic

fibration with singularities. Take \( p \) near \( c_0 \), then the "general

fiber" \( M := W^{-1}(p) \) is a symplectic submanifold.

Today: How are these two related?

Recall \( \text{ob } F(E, W) = W \)-admissible lagrangians,

where \( D_L \subset \mathbb{R} \) the set of heights. On cohomology,

\[
\hom_{F(E, W)} (K, L) = \text{HF}^* (\phi_e^* K, L)
\]

("sufficiently large admissible positive flow"

The cohomology of the other definition (with oriented stuff)
always coincides with this.
Idea: can define functors, which are adjoint:

\[ \phi_{2n} : \mathcal{F}(E, w) \xrightarrow{\sim} \mathcal{F}(w) \]

\( n \) is "cap", some intersection.

\( \varphi_{cW} \) and \( \varphi_{ew} \) are the Ozsváth functors, counterclockwise & clockwise

\( \rho \) is the monodromy, induced by a transport around large loop.

\( \phi_{2n} \) is "once wrapping, counterclockwise".

In general, there is a \( \forall \alpha \in \mathbb{Z} \)

**Proposition:** [Kontsevich, Seidel] \( \phi_{2n} \) is the "Serre functor", up to degree shift.

* \( \mathcal{O} : \mathcal{F}(E, w) \to \mathcal{F}(M) \) is "intersection with a fiber" ([Abouzaid-Seidel, Abouzaid-Groves]).

  - If \( L \) has 1 horizontal end, \( \mathcal{O} : L \to \mathcal{O}L := L \cap W^x(p) \)
  
  \[ \Delta \to V = \partial \Delta \text{ vanishing cycle} \]

  - If \( L \) has multiple ends, we should think of \( \mathcal{O} \) as bending in \( \text{Tor} \mathcal{F}(M) \), \( \text{Perf} \mathcal{F}(M) \),

\[ \mathcal{O}L \xrightarrow{\Delta} \text{Cone} \left( \tau^*L \to \tau^*L \right), \]

where \( \tau \) is some morphism coming from count of discs in the total space

[cf. Biran-Cornea's work on larger cobordisms]
* $U_{cw} : F(M) \rightarrow F(E,W)$ "Curl functor" : $L \mapsto U_{L}^{\eta}$

$U_{L}^{\eta} = \{ \text{pts in } W^\eta(\text{any}) \text{ which all transport to } L \text{ along } \eta \}$

Note: $U_{cw} = U_{cw} \circ \mu^{-1}$, so let's not talk about $U_{cw}$.

These structures are related; for instance

**Proposition**: (Abazawa-Ganatra) up to a degree shift (by 2?):

1. $U_{L} = \phi_{\eta L}$
2. $\mu \circ \phi_{\eta L} \circ U_{L} = \mu_{\eta L}$

**Theorem**: (Abazawa-Ganatra) there are exact triangles of functors:

1. In $F(E,W)$, $id \rightarrow \phi_{\eta L}$, meaning $YL \in \text{ob } F(E,W)$,

   there is a triangle $L \rightarrow \phi_{\eta L}$

2. In $F(M)$, $id \rightarrow \mu$, meaning $YK \in \text{ob } F(M)$, there is

   a triangle $K \rightarrow \mu K$

Note by definition: $NUK \in \text{some Corr } (K \rightarrow \mu K)$.

**Example application**: (LES of a Dehn twist)

Given $L \in M$ a (parametrized) Lagrangian sphere in $M$, there is a Lagrange fibration $(E,W)$ with one critical point, with fiber $M$ and with vanishing cycle $= L$.

The Dehn twist $T_{L}$ can be defined to be (up to Hamiltonian isotopy) the monodromy $\mu : M \rightarrow M$ associated to $(E,W)$. 
Or, define $\tau_L$ directly as an element of $\text{Symp}^\times \hat{(E)}$, and hence as an element of $\text{Symp}^\times (M)$ (by existence of Weinstein neighborhoods). $\tau_L$ is the antipodal map on $L$, and the identity far from $L$.

We know, for this $(E, w)$:

- **Theorem.** [Seidel, Abelian Generation] the triangle $\Delta$ generates $F(E, w)$, and moreover $\text{End}_F(\Delta) \cong k$.

As a consequence, if $k \in \text{ob} F(\Delta)$ is arbitrary, then

$$UK \cong \text{Hom}_F(\Delta, UK) \otimes \Delta$$

is an object of $F(E, w)$. Let $\phi : UK \rightarrow L \otimes K$ be a projection of $UK$ onto the category split-generated by $\Delta$.

$$\Delta$$

intersects exactly at $L \otimes K$.

$$\Delta$$

And $UK \cong \text{CF}^\bullet(L, K) \otimes \Delta$.

Hence, the tautological exact triangle

$$K \rightarrow \tau_L K \rightarrow L \otimes K$$

(can be projected by $UK$), which is

$$K \rightarrow \tau_L K \rightarrow \text{CF}^\bullet(L, K) \otimes L$$

which realizes the LES of a Dehn twist [Seidel].

Need to do (for instance): build a functor

$$F(E, w) \rightarrow F(M)$$
First, we will give a different construction of \( F(M) \), which is more manifestly compatible with Abouzaid-Seidel's \( F(E,W) \) (cf. Abouzaid-Seidel), essentially using \( \mathcal{L} \).

Let \( X \) be a symplectic manifold.

* Fix a set of objects \( S = \text{ob} \mathcal{F}(X) \) of Lagrangian branes.

For technical reasons, assume \( S \) is countable.

* For each \( L \in S \) and \( i \in \mathbb{N} \), pick a Thom perturbation \( L^{(i)} \), satisfying:

  - For any finite subset \( \{ L_{\alpha_0}^{(k_0)}, \ldots, L_{\alpha_d}^{(k_d)} \} \) with all \( k \) distinct, these are in general position (pairwise transverse) (except possibly for \( k_0 \gg k_d \)).

  (easy to do by induction, by countability)

Define a directed \( \mathcal{A}_{\infty} \)-category \( \mathcal{O}_M \) as follows:

* \( \text{ob} \mathcal{O}_M := \{ L^{(k)} \mid L \in S, \ k \in \mathbb{N} \} \) (or rather pairs \( (L,k) \), so that \( \mathcal{O}_M \) remembers the integer chosen)

* \( \text{hom}_{\mathcal{O}_M} (L_{\alpha}^{(k_0)}, L_{\beta}^{(k_1)}) = \begin{cases} \mathbb{C}^* & \text{if } k_0 > k_1 \text{ or } k_1 = k_0 \text{ and } L_{\alpha} = L_{\beta} \\ 0 & \text{otherwise} \end{cases} \)

* the \( \mathcal{A}_{\infty} \)-structure maps only non-zero for \( k_0 > k_1 > k_2 \).

(Case of equality: strict unitarity) Moreover, in this case, \( \{ L_i^{(k)} \}_{k \geq 0} \) are pairwise transverse, so in the court of discs, we can set the Hamiltonian term to be equal to zero. So we are genuinely counting \( J \)-holomorphic disks, albeit with potentially domain-dependent \( J \) in higher dimensions.

\[ \text{ex. } T^2 / \mathbb{Z} \]

In the Riemann surface case, \( \mathcal{O}_M \) is essentially combinatorial, though there are infinitely many objects.
Define $Z_n \in H^0(\Omega_M)$ the "quasi-units".

\[ q \in \text{hom}_{\Omega_M}^\circ (K^{(s)}, K^{(t)}) \quad \text{whenever } s > t \]

\[ \text{where } K^{(s)} = \text{def by eq) continuation maps.} \]

\[ \text{CE}^0((K^{(s)}, K^{(t)})) \quad \text{for } s > t \]

\[ q = \bigoplus_{x \in \mathbb{M}} (\mathbb{K}(x) \otimes K^{(t)})_x \]

These are the elements that should induce isomorphisms

\[ K^{(s)} \cong K^{(t)} \quad \text{in } \mathbb{F}(M) \]

Ca Define $F^{loc}_M = \Omega_M \otimes_{\mathbb{Z}} (L^{-1}_M)$. Have $j : \Omega_M \to F^{loc}_M$

**Proposition:** there is a quasi-equivalence $F^{loc}_M \cong \mathbb{F}(M)$, where

$\mathbb{F}(M)$ is the Fukaya category with objects $I$.

**Remark:** in the exact case, if $K, K'$ hom. isotopic in $\mathbb{F}(M)$,

\[ \text{hom}_M(K, K') \cong \text{hom}_M(K, K) \cong H^1_0(K) \]

\[ j^\circ (-, q) : \text{hom}_M(L, K) \cong \text{hom}_M(L, K') \]

**Lemma (analogue of "correct position lemma") If } s > t, \text{ then**

\[ j^\circ : \text{hom}_{\Omega_M}(K^{(s)}, L^{(t)}) \to \text{hom}_{F^{loc}_M}(K^{(s)}, L^{(t)}) \]

In particular, $\text{hom}_{F^{loc}_M}(K^{(s)}, K^{(t)}) \cong \text{hom}_{\Omega_M}(K^{(s)}, K^{(t)})$

\[ \text{for } K \text{ lemma} \]

\[ \text{hom}_{\Omega_M}(K^{(s)}, K^{(t)}) \cong \text{CE}^0((K^{(s)}, K^{(t)})) \]

**Remark:** a $K^{(s)}$ in $\mathbb{F}(M)$ corresponds to $K^{(s)} \rightarrow K^{(s)} \rightarrow K^{(s)} \rightarrow \ldots$ in $\Omega_M$.

**Sketch of proof of Prop.**

Define $F^{bg}_M$ a version of Fukaya category defined as usual, with objects $\{L^{(s)} | L \in \Sigma, \text{ i.e } \mathbb{N} \cup \{L | \exists \leq 2\}$. 


* We can choose the perturbations involved in \( \mu^d \) to agree on each \( \mathfrak{m} \) : to agree with \( F(\mathfrak{m}) \) when all objects are in \( \{ \mathcal{L}(\mathcal{C}, \mathcal{B}) \} \).

\[ \mathcal{F}(\mathfrak{m}) \sim \mathcal{F}^\text{big}(\mathfrak{m}) \]. Note that this is a quasi-equivalence:

each \( \mathcal{L}^{(1)} \sim \mathcal{L} \) for \( \mathcal{L} \in \mathfrak{m} \).

* We can also choose perturbations such that for \( (\mathcal{L}_0^{(k_0)}, \ldots, \mathcal{L}_d^{(k_d)}) \) with \( k_0 \rightarrow \cdots \rightarrow k_d \), agree with choices made in \( \mathcal{O}_\mathfrak{M} \). This essentially gives \( \exists \mathcal{F} : \mathcal{O}_\mathfrak{M} \rightarrow \mathcal{F}^\text{big}_\mathfrak{M} \) a strict functor (i.e., \( \mathcal{F}^d \) are zero for \( d > 1 \)). We have to worry a bit about strict units, but let's not.

* Note that in \( \mathcal{F}^\text{big}_\mathfrak{M} \), morphisms in \( \mathfrak{M}^d \) are sent to isomorphisms.

By universal property,

\[ \mathcal{O}_\mathfrak{M} \xrightarrow{} \mathcal{F} \xrightarrow{} \mathcal{F}^\text{big}_\mathfrak{M} \]

\[ \mathcal{F} \xrightarrow{} \mathcal{F}^\text{big}_\mathfrak{M} \]

* All we need to check is that \( \mathcal{F} \) is cohom. fully faithful, because then

\[ \mathcal{F}^\text{big}_\mathfrak{M} \xrightarrow{} \mathcal{F}^\text{big}_\mathfrak{M} \]

by inverting \( \sim \)-equiv.

\[ \mathcal{F}(\mathfrak{m}) \]

* Cohom. fully faithfulness of \( \mathcal{F} : \mathfrak{m} \rightarrow \mathcal{F}^\text{big}_\mathfrak{M} \):

\[ \text{hom}_\mathcal{F}(\mathcal{K}^{(s)}, \mathcal{L}^{(r)}) \sim \text{hom}_\mathcal{F}^\text{big}(\mathcal{K}^{(s+\mathfrak{N})}, \mathcal{L}^{(r)}) \]

because \( \mathcal{K}^{(s)} \subset \mathcal{K}^{(s+\mathfrak{N})} \),

\[ \mathcal{F}(\mathfrak{m}) \xrightarrow{} \mathcal{F}^\text{big}_\mathfrak{M} \]

by exactness lemma \( \rightarrow j_\beta \) with \( \mathfrak{N} \) big enough so that \( s + \mathfrak{N} \geq t \)

\[ \text{hom}_\mathcal{F}^\text{big}(\mathcal{K}^{(s)}, \mathcal{L}^{(r)}) \xrightarrow{} \text{CF}^\text{big}(\mathcal{K}^{(s+\mathfrak{N})}, \mathcal{L}^{(r)}) \]

(just chain-level bijection)