25/05/16: Rem: about $UK \simeq \hom_{\mathcal{F}}(\Delta, UK) \otimes \Delta$. For $\mathcal{A} \overset{i}{\subseteq} \mathcal{E}$ such that $\mathcal{A}$ generates $\mathcal{E}$, then any object $K \in ob\,\mathcal{E}$ is isomorphic to $i_* i^* K$, for adjoint functors $Mod(\mathcal{A}) \underset{i^*}{\overset{i_*}{\rightleftarrows}} Mod(\mathcal{E})$.

* Analogy: $(Cat, \mathcal{E}^{op} \times \mathcal{E} \xrightarrow{\hom(-,-)} Ch_{\Bbbk}) \longleftrightarrow (Vect, \text{inner product } V \times V \xrightarrow{\langle -,- \rangle} \Bbbk)$.

  $W \overset{i}{\subseteq} V$ map of v.s. with inner product. We have a projection $i_* i^* : V \to V$. And $W = V \iff i_* i^* = id$.

* Given an object $K \in ob\,\mathcal{E}$, there is a canonical module $i_* i^* K \in Mod(\mathcal{E})$,

(§) $$i_* i^* K(X) := \hom_{\mathcal{E}}\Big(X, \underbrace{\bigoplus_{A_0, \dots, A_s \in ob\,\mathcal{A}} \hom_{\mathcal{E}}(A_s, K) \otimes \hom_{\mathcal{A}}(A_{s-1}, A_s) \otimes \dots \otimes \hom_{\mathcal{A}}(A_0, A_1) \otimes A_0}_{\text{"} i_* i^* K \text{"}}\Big)$$

rather Y_{A_0}

Proposition: $\mathcal{A}$ split-generates $\mathcal{E}$ $\iff i_* i^* K \simeq K$

$\iff K$ is a summand of $i_* i^* K$

$\iff 1 = 0$ in $\hom_{\mathcal{E}/\mathcal{A}}(K,K) := Cone\big(\hom_{\mathcal{E}}(K, i_* i^* K) \xrightarrow{\mu} \hom_{\mathcal{E}}(K,K)\big)$

Now, in the setting where one thimble Δ generates, let $\mathcal{A}$ be the subcategory with one thimble. There is a "minimal model" $\mathcal{A} \overset{q\text{-equiv}}{\simeq} \mathcal{A}_{min}$ with $ob\,\mathcal{A}_{min} = \{\Delta\}$, $End_{\mathcal{A}_{min}}(\Delta) = \Bbbk\langle e_\Delta^+ \rangle$.

Lemma: when $\mathcal{A}$ is strictly unital and augmented (ie $\hom_{\mathcal{A}}(X,X) \xrightarrow{f_X} \Bbbk$, $e_X^+ \mapsto 1$), define the augmentation ideal by

$$\hom_{\overline{\mathcal{A}}}(X,Y) = \begin{cases} \hom_{\mathcal{A}}(X,Y) & \text{if } X \neq Y \\ \ker(f_X) & \text{if } X = Y \end{cases}$$

(so, it is everything but the e_X^+'s). The lemma is that there is a complex, quasi-isomorphic to (§), that consists in (§) with all the $\mathcal{A}$'s replaced by $\overline{\mathcal{A}}$.

↳ For our $\mathcal{A}_{min} \subseteq \mathcal{F}(E,W)$ given by the thimble Δ, we get

$i_* i^* K = \hom_{\mathcal{F}}(\Delta, K) \otimes \Delta$, and no other terms.

Last time: (E,W) LG model, M fiber.

We discussed functors (at least cohomologically)

$$\phi_{2\pi} \circlearrowright \mathcal{F}(E,W) \underset{\cup}{\overset{\cap}{\rightleftarrows}} \mathcal{F}(M) \circlearrowleft \mu, \quad \text{with 2 exact triangles}$$

- In $\mathcal{F}(E,W)$: $\begin{tikzcd} id \arrow[r] & \phi_{2\pi} \arrow[dl] \\ \cup\cap \arrow[u] \end{tikzcd}$
- In $\mathcal{F}(M)$: $\begin{tikzcd} id \arrow[r] & \mu \arrow[dl] \\ \cap\cup \arrow[u] \end{tikzcd}$

Let's understand the first one more.

More on $\phi_{2\pi}$: "once wrapping". There is also $\phi_{2\pi k}$, $k \in \mathbb{Z}$.
We saw that, at least cohomologically, $\phi_{-2\pi}$ is the Serre functor, up to a shift by n.

Definition: $\mathcal{E}$ category. A Serre functor (shifted by n) is an automorphism $S: \mathcal{E} \to \mathcal{E}$ such that $\forall X, Y$, there is a cohomological perfect pairing $\hom^*_{\mathcal{E}}(SX, Y) \otimes \hom(Y,X) \to \mathbb{k}[-n]$.

So, there is an isomorphism $\mathrm{Hom}^*(SX,Y) \cong \mathrm{Hom}^{n-*}(Y,X)^\vee$

ex: V proper smooth algebraic variety over $\mathbb{C}$; then $-\otimes \omega_V$ is a Serre functor for $\mathrm{Coh}(V)$: Serre duality implies $\mathrm{Ext}^*(\mathcal{E} \otimes \omega_V, \mathcal{F}) \cong \mathrm{Ext}^{n-*}(\mathcal{F}, \mathcal{E})^\vee$.

$$\Lambda^{top} T^* V \quad \text{``} \omega_V$$

If V is Calabi-Yau, then (by def) ω_V is trivial $\cong \mathbb{C}$, so the Serre functor is trivial (up to shift by n).
"$\mathrm{Coh}(V)$ is a CY category of dim n": $\mathrm{Hom}^*(X,Y) \cong \mathrm{Hom}^{n-*}(Y,X)$.

Symplectic setting:

Note that when K, L are compact Lagrangians, or non-compact Lagrangians which are $\pitchfork$ near ∞ (so one can define $CF^*(-,-)$), then note that $HF^*(K,L) \cong HF^{n-*}(L,K)^\vee$.

If $K \pitchfork L$, then $CF^*(K,L)$ and $CF^*(L,K)$ both have the same generators, namely $\{K \pitchfork L\}$.